Real Analysis

Math 127A-B, Winter 2019

Midterm 1: Solutions

1. [15pts] State three different properties that characterize the completeness of the real numbers $\mathbb{R}$.

Solution.

Any three from the following.

- Every nonempty set bounded from above has a supremum (Dedekind complete).
- Every bounded, increasing (or monotone) sequence converges (monotone convergence).
- Every Cauchy sequence converges (Cauchy complete).
- Every nested sequence of nonempty, closed, bounded intervals has nonempty intersection (nested interval property).
- Every bounded sequence has a convergent subsequence (Bolzano-Weierstrass).

Note that $\mathbb{R}$ satisfies all of these properties but $\mathbb{Q}$ does not satisfy any of them.

2. [20pts] Say if the following statements are true or false. If true, give a brief explanation (a complete proof is not required); if false, give a counterexample.

(a) If $a, b \in \mathbb{R}$ and $a < b + 1/n$ for every $n \in \mathbb{N}$, then $a \leq b$.

(b) Every bounded monotonic sequence is a Cauchy sequence.

(c) If the sequences (x_n) and (y_n) diverge, then the sequence $(x_n y_n)$ diverges.

(d) If a sequence (x_n) contains a subsequence (x_{n_k}) such that $x_{n_k} \to \infty$ as $k \to \infty$, then $x_n \to \infty$ as $n \to \infty$.

Solution

- (a) True. Take the limit of $a < b + 1/n$ as $n \to \infty$ and use the order properties of limits.
- (b) True. Every bounded monotone sequence converges (by the monotone convergence theorem) and every convergent sequence is Cauchy.
- (c) False. For example, if $x_n = y_n = (-1)^{n+1}$, then (x_n) and (y_n) diverge but $(x_n y_n)$ converges to 1.
- (d) False. For example, if $x_n = (-1)^{n+1} n$ and $n_k = 2k - 1$, then $x_{n_k} = 2k - 1 \to \infty$, but (x_n) does not diverge to ∞ because there are infinitely many terms such that $x_n < 0$.

3. [20pts] (a) State the definition of the convergence of a sequence (x_n) of real numbers to a limit x.

(b) Let (x_n) and (y_n) be sequences of real numbers, and let (z_n) be the "shuffled" sequence $(x_1, y_1, x_2, y_2, \ldots)$ defined by $z_{2k-1} = x_k$, $z_{2k} = y_k$. Prove that (z_n) converges to $x \in \mathbb{R}$ if and only if both (x_n) and (y_n) converge to x.

Solution

- (a) A sequence (x_n) converges to $x \in \mathbb{R}$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for every $n > N$.

- (b) First, suppose that (x_k) and (y_k) both converge to $x \in \mathbb{R}$. Then given any $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $|x_k - x| < \epsilon$ for all $k > N_1$, and $|y_k - x| < \epsilon$ for all $k > N_2$. Choose $N = 2\max\{N_1, N_2\}$. We claim that $n > N$ implies $|z_n - x| < \epsilon$, which proves that $z_n \to x$ as $n \to \infty$.

- To prove the claim, note that if $n = 2k - 1 > N$ is odd, then $k > (N+1)/2 > N_1$, so $|z_n - x| = |x_k - x| < \epsilon$, and if $n = 2k > N$ is even, then $k > N_2$, so $|z_n - x| = |y_k - x| < \epsilon$.

- (b) Conversely, suppose that (z_n) converges to $x \in \mathbb{R}$. Then (x_k) and (y_k), both converge to x, since they are subsequences of (z_n) and every subsequence of a convergent sequence converges to the limit of the sequence. (Or you can give an ϵ-N proof.)

4. [20pts] (a) Let (x_n) be a bounded sequence of real numbers. Define $\limsup_{n\to\infty} x_n$.

(b) Suppose that (x_{n_k}) is a convergent subsequence of a bounded sequence (x_n) of real numbers. Prove that

$$\lim_{k\to\infty} x_{n_k} \le \limsup_{n\to\infty} x_n.$$

Solution

- (a) If (x_n) is a bounded sequence, then

  $$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} y_n \qquad y_n = \sup\{x_k : k \ge n\}.$$

  The supremum exists since the set $\{x_k : k \ge n\}$ is nonempty and bounded, while the limit exists since the sequence (y_n) is decreasing and bounded.

- (b) Let (x_{n_k}) be a convergent subsequence of a bounded sequence (x_n), with $x_{n_k} \to x$ as $k \to \infty$. For every $k \in \mathbb{N}$, we have

  $$x_{n_k} \le y_{n_k}. \tag{1}$$

  Then (y_{n_k}) is a subsequence of the convergent sequence (y_n), so

  $$y_{n_k} \to y = \limsup_{n\to\infty} x_n \qquad \text{as } k \to \infty.$$

  Taking the limit of (1) as $k \to \infty$ and using the order properties of limits, we get that $x \le y$, which proves the result.