Real Analysis
Math 127A-B, Winter 2019
Midterm 2

1. [15pts] (a) Give the definition of an open set $G \subset \mathbb{R}$.
(b) Give two definitions of a closed set $F \subset \mathbb{R}$, one in terms of open sets, the other in terms of sequences.

Solution

- (a) A set $G \subset \mathbb{R}$ is open if for every $x \in G$ there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset G$.
- (b) (i) A set $F \subset \mathbb{R}$ is closed if $F^c = \mathbb{R} \setminus F$ is open. (ii) A set $F \subset \mathbb{R}$ is closed if the limit of every convergent sequence in F belongs to F.

2. [20pts] Say if the following statements are true or false. If true, give a brief explanation (a complete proof is not required); if false, give a counterexample.

(a) The series

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^2}+\frac{(-1)^{n+1}}{n}\right]$$

converges conditionally.

(b) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ converges.

(c) If $A \subset \mathbb{R}$ is not closed, then A is open.

(d) If every $a \in A$ is a limit point of $A \subset \mathbb{R}$, then A is closed.

Solution

- (a) True. The series converges since

$$\sum_{n=1}^{\infty}\left[\frac{1}{n^2}+\frac{(-1)^{n+1}}{n}\right]=\sum_{n=1}^{\infty}\frac{1}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

  is the sum of two convergent series. The series does not converge absolutely by the comparison test, since

$$\left|\frac{1}{n^2}+\frac{(-1)^{n+1}}{n}\right| \geq \frac{1}{2n} \qquad \text{for } n \in \mathbb{N},$$

  and the harmonic series diverges.

- (b) True. Since $\sum a_n$ converges, we have $a_n \to 0$ as $n \to \infty$, so there exists $N \in \mathbb{N}$ such that $|a_n| \leq 1$ and $0 \leq a_n^2 \leq |a_n|$ for all $n > N$. Then comparison with the convergent series $\sum |a_n|$ implies that $\sum a_n^2$ converges absolutely.

- (c) False. For example $A = (0, 1]$ is not open, since no neighborhood of $1 \in A$ is contained in A, and not closed since the sequence $(1/n)$ in A converges to 0 and $0 \notin A$.

- (d) False. For example, every $r \in \mathbb{Q}$ is a limit point of $\mathbb{Q}$, since every neighborhood of r contains rational numbers distinct from r, but $\mathbb{Q}$ is not closed since there is a convergent sequence of rational numbers with an irrational limit that does not belong to $\mathbb{Q}$. (What is true is that a set A is closed if every limit point $x \in \mathbb{R}$ belongs to A.)

3. [20pts] (a) Define what it means for a series $\sum_{k=1}^{\infty} a_k$ to converge.
(b) Suppose that $\sum_{k=1}^{\infty} a_k$ is a convergent series. Prove that $\sum_{k=n+1}^{\infty} a_k$ converges for every $n \in \mathbb{N}$, and show that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=n+1}^{\infty} a_k \right| < \epsilon \qquad \text{for all } n > N.$$

Solution

- (a) A series $\sum_{k=1}^{\infty} a_k$ converges to $S \in \mathbb{R}$ if $S_n \to S$ as $n \to \infty$, where $S_n = \sum_{k=1}^{n} a_k$ is the nth partial sum of the series. That is, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{n} a_k - S \right| < \epsilon \qquad \text{for all } n > N.$$

- (b) Let $S_m = \sum_{k=1}^{m} a_k$, so that $S_m \to S = \sum_{k=1}^{\infty} a_k$ as $m \to \infty$. Then

$$\sum_{k=n+1}^{m} a_k = S_m - S_n \qquad \text{for } m > n.$$

Taking the limit of this equation as $m \to \infty$ with n fixed, we get that

$$\sum_{k=n+1}^{\infty} a_k = \lim_{m \to \infty} (S_m - S_n) = S - S_n,$$

so the series converges.

- In addition, since $S_n \to S$ as $n \to \infty$, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|S - S_n| < \epsilon$ for all $n > N$, meaning that

$$\left| \sum_{k=n+1}^{\infty} a_k \right| < \epsilon \qquad \text{for all } n > N.$$

4. [20pts] (a) Define what it means for $x \in \mathbb{R}$ to be a limit point of $A \subset \mathbb{R}$.
(b) Suppose that $A \subset \mathbb{R}$ is nonempty, bounded from above, and $\sup A \notin A$. Prove that $\sup A$ is a limit point of A.

Solution

- (a) A real number $x \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$ if for every $\epsilon > 0$ there exists $a \in A$ with $a \neq x$ such that $a \in (x - \epsilon, x + \epsilon)$. Equivalently, $x \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$ if there is a sequence (a_n) of points $a_n \in A$ with $a_n \neq x$ such that $a_n \to x$ as $n \to \infty$.

- (b) Let $\epsilon > 0$. Since $\sup A$ is the least upper bound of A, there exists $a \in A$ such that $\sup A - \epsilon < a \leq \sup A$, and $a \neq \sup A$ since $\sup A \notin A$. It follows that for every $\epsilon > 0$, there exists $a \in A$ with $a \neq \sup A$ such that $a \in (\sup A - \epsilon, \sup A + \epsilon)$, meaning that $\sup A$ is a limit point of A.