Density of the Rationals

An essential property of the natural numbers is the following induction principle, which expresses the idea that we can reach every natural number by counting upwards from one.

Axiom 1. Suppose that $A \subset \mathbb{N}$ is a set of natural numbers such that: (a) $1 \in A$; (b) $n \in A$ implies $(n+1) \in A$. Then $A = \mathbb{N}$.

Before stating the next theorem, we give a formal definition of the maximal and minimal elements of a set.

Definition 2. Suppose that $A \subset \mathbb{R}$ is a set of real numbers. A maximal element of A is a number $M = \max A$ such that $M \in A$ and $M \geq x$ for every $x \in A$. A minimal element of A is a number $m = \min A$ such that $m \in A$ and $m \leq x$ for every $x \in A$.

It follows immediately from the definition that if A has a maximal or minimal element, then $\sup A = \max A$ or $\inf A = \min A$. However, $\sup A$ or $\inf A$ may exist, even when $\max A$ or $\min A$ don't exist, if they don't belong to A.

The following result states a fundamental property of the natural numbers called the well-ordering property.

Theorem 3. Every nonempty subset of natural numbers contains a minimal element.

Proof. We will use induction to show that if a set $S \subset \mathbb{N}$ contains no minimal element, then S is the empty set, which proves the result.

Let

$$A = \{n \in \mathbb{N} : 1, 2, \ldots, n \notin S\}.$$

If $1 \notin A$, then $1 \in S$, so 1 a minimal element of S since $1 \leq n$ for every $n \in \mathbb{N}$, which contradicts our assumption on S. Hence, $1 \in A$.

Suppose that $n \in A$ for some $n \in \mathbb{N}$, meaning that $1, 2, \ldots, n \notin S$. If $n+1 \notin A$, then $n+1 \in S$, so $n+1$ is a minimal element of S. This contradiction shows that $n+1 \in A$. It follows by induction that $A = \mathbb{N}$, which implies that $S = \varnothing$. □

One can also prove that the well-ordering property of $\mathbb{N}$ implies the induction axiom, so the well-ordering of $\mathbb{N}$ is equivalent to induction.

The integers $\mathbb{Z}$ are not well-ordered (for example, $\mathbb{Z}$ itself has no minimal element), but a similar property holds for sets of integers that are bounded from below.

Corollary 4. If a nonempty set $S \subset \mathbb{Z}$ is bounded from below in $\mathbb{R}$, then S has a minimal element.

Proof. Since S is bounded from below in $\mathbb{R}$, the Archimedean property of $\mathbb{R}$ implies that S is bounded from below by some integer $a \in \mathbb{Z}$. Then

$$B = \{n - a + 1 : n \in S\} \subset \mathbb{N},$$

so Theorem 3 implies that B has a minimal element $b \in B$. It follows that $m = b + a - 1 \in S$ is a minimal element of S. $\square$

We can now prove the following (intuitively obvious) lemma, which we will use to prove the density of $\mathbb{Q}$ in $\mathbb{R}$.

Lemma 5. If $x, y \in \mathbb{R}$ satisfy $y - x > 1$, then there exists $m \in \mathbb{Z}$ such that $x < m < y$.

Proof. We construct m as the minimal integer strictly greater than x. Let

$$S = \{n \in \mathbb{Z} : x < n\}.$$

Then $S \subset \mathbb{Z}$ is nonempty (by the Archimedean property) and bounded from below (by x), so it has a minimal element $m \in S$ with $m - 1 \notin S$. It follows that $x < m$ and $x \geq m - 1$, so $x < m \leq x + 1 < y$, which proves the result. $\square$

Finally, we prove the density of the rational numbers in the real numbers, meaning that there is a rational number strictly between any pair of distinct real numbers (rational or irrational), however close together those real numbers may be.

Theorem 6. If $x, y \in \mathbb{R}$ and $x < y$, then there exists $r \in \mathbb{Q}$ such that $x < r < y$.

Proof. Let $\epsilon = y - x > 0$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $0 < 1/n < \epsilon$, which implies that $ny - nx > 1$. By Lemma 5, there exists $m \in \mathbb{Z}$ such that $nx < m < ny$, which proves the result with $r = m/n$. $\square$

Repeated application of this theorem shows that there are, in fact, infinitely many rational numbers between any pair of distinct real numbers. One can also use the result to prove that the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.