Summary of Topics: Real Analysis (127A)

1 Real numbers

Axiom 1 (Axioms for $\mathbb{R}$)**.** The real numbers $\mathbb{R}$ are a Dedekind complete, ordered field.

Ordered field means that $\mathbb{R}$ is equipped with an order relation $<$ together with addition and multiplication operations $+$ and $\cdot$, with their usual order and algebraic properties.

Dedekind complete means that every nonempty set $A \subset \mathbb{R}$ that is bounded from above has a supremum $\sup A \in \mathbb{R}$.

Definition 2 (Supremum and infimum)**.** Suppose $A \subset \mathbb{R}$. Then $\sup A \in \mathbb{R}$ is the supremum of A if it is the least upper bound of A, i.e. $x \leq \sup A$ for every $x \in A$, and if $M < \sup A$, then there exists $x \in A$ such that $x > M$. Similarly, $\inf A \in \mathbb{R}$ is the infimum of A if it is the greatest lower bound of A, i.e. $x \geq \inf A$ for every $x \in A$, and if $m > \inf A$, then there exists $x \in A$ such that $x < m$.

The extended real numbers are $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, ordered in the obvious way. We define $\sup A = \infty$ if A isn't bounded from above, $\inf A = -\infty$ if A isn't bounded from below, and $\sup \varnothing = -\infty$, $\inf \varnothing = \infty$. Then the supremum and infimum of every subset of $\mathbb{R}$ is defined as an extended real number.

The following theorems are a consequence of the completeness of $\mathbb{R}$.

Theorem 3 (Archimedean property)**.** For every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $x < n$, and for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $0 < 1/n < \epsilon$.

Theorem 4 (Density of rationals)**.** If $x, y \in \mathbb{R}$ and $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

Theorem 5 (Nested interval property)**.** If $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ is a nested sequence of nonempty, closed, bounded intervals $I_n = [a_n, b_n]$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

2 Sequences

Definition 6 (Sequences)**.** A sequence (x_n) (of real numbers) is an ordered list of real numbers $x_n \in \mathbb{R}$, indexed by $n \in \mathbb{N}$. Equivalently, a sequence (x_n) is defined by a function $f : \mathbb{N} \to \mathbb{R}$ where $x_n = f(n)$.

Definition 7 (Convergence of a sequence)**.** A sequence (x_n) converges to a limit $L \in \mathbb{R}$, written $\lim_{n\to\infty} x_n = L$ or $x_n \to L$ as $n \to \infty$, if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for every $n > N$.

We sometimes omit the condition "$n \to \infty$," which is understood from the definition of the limit of a sequence.

Definition 8 (Divergence to $\pm\infty$)**.** A sequence (x_n) diverges to ∞, written $\lim x_n = \infty$ or $x_n \to \infty$, if for every $M \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n > M$ for every $n > N$. Similarly, (x_n) diverges to $-\infty$, written $\lim x_n = -\infty$ or $x_n \to -\infty$, if for every $m \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n < m$ for every $n > N$.

Theorem 9 (Squeeze)**.** If $a_n \le x_n \le b_n$ and $a_n \to L$, $b_n \to L$, then $x_n \to L$.

Definition 10 (Bounded sequence)**.** A sequence (x_n) is bounded if there exists $M \ge 0$ such that $|x_n| \le M$ for every $n \in \mathbb{N}$.

Theorem 11. A convergent sequence is bounded.

Corollary 12 (Divergence criterion)**.** An unbounded sequence diverges.

Theorem 13 (Algebraic and order properties)**.** Suppose that (x_n) and (y_n) are convergent sequences with $x_n \to L$ and $y_n \to M$. Then: (i) $kx_n \to kL$ for any $k \in \mathbb{R}$; (ii) $x_n + y_n \to L + M$; (iii) $x_n y_n \to LM$; (iv) $x_n/y_n \to L/M$ provided that $M \neq 0$; (v) if $x_n \le y_n$ for every $n \in \mathbb{N}$, then $L \le M$.

Definition 14 (Monotone sequences)**.** A sequence (x_n) is increasing if $x_{n+1} \ge x_n$ for every $n \in \mathbb{N}$, decreasing if $x_{n+1} \le x_n$ for every $n \in \mathbb{N}$, and monotone if it is increasing or decreasing.

Theorem 15 (Monotone convergence)**.** A monotone sequence converges if and only if it is bounded. An unbounded increasing sequence diverges to ∞, and an unbounded decreasing sequence diverges to $-\infty$.

Definition 16 (Cauchy sequences)**.** A sequence (x_n) is Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon$ for all $m, n > N$.

Theorem 17 (Cauchy criterion)**.** A sequence converges if and only if it is Cauchy.

Definition 18 ($\limsup$ and $\liminf$)**.** If (x_n) is a bounded sequence, then

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \ge n\}, \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \ge n\}.$$

The $\limsup$ and $\liminf$ of a bounded sequence are well-defined as real numbers. One can also define the $\limsup$ and $\liminf$ of unbounded sequences as extended real numbers.

Theorem 19 (Convergence criterion for $\limsup$ and $\liminf$)**.** A sequence (x_n) converges if and only if $\limsup x_n = \liminf x_n \in \mathbb{R}$.

Definition 20 (Subsequence)**.** A subsequence of $(x_n)_{n=1}^{\infty}$ is a sequence $(x_{n_k})_{k=1}^{\infty}$ where $n_1 < n_2 < \cdots < n_k < \cdots$ is a strictly increasing sequence of indices.

Theorem 21 (Convergence of subsequences)**.** Every subsequence of a convergent sequence converges to the same limit as the sequence.

Corollary 22 (Divergence criterion)**.** If a sequence has a subsequence that diverges or has two subsequences that converge to different limits, then the sequence diverges.

Theorem 23 (Bolzano-Weierstrass)**.** Every bounded sequence has a convergent subsequence.

3 Series

Definition 24 (Convergence of a series)**.** A series $\sum_{n=1}^{\infty} a_n$ converges to a sum $S \in \mathbb{R}$, written $\sum_{n=1}^{\infty} a_n = S$, if the sequence (S_n) of partial sums $S_n = \sum_{k=1}^{n} a_k$ converges to S. That is, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{n} a_k - S \right| < \epsilon \qquad \text{for every } n > N.$$

A series diverges to $\pm\infty$ if the sequence of partial sums diverges to $\pm\infty$.

Definition 25 (Absolute and conditional convergence)**.** A series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ converges conditionally if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem 26. An absolutely convergent series converges.

Theorem 27. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim a_n = 0$.

Corollary 28 (Divergence criterion)**.** If the sequence (a_n) doesn't converge to 0, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 29 (Geometric series)**.** Let $a \in \mathbb{R}$. The geometric series $\sum_{n=0}^{\infty} a^n$ converges absolutely if $|a| < 1$, in which case $\sum_{n=0}^{\infty} a^n = 1/(1-a)$, diverges to ∞ if $a \geq 1$, and diverges if $a \leq -1$.

Theorem 30 (p-series)**.** Let $p > 0$. The series $\sum_{n=1}^{\infty} 1/n^p$ converges absolutely if $p > 1$ and diverges to ∞ if $0 < p \leq 1$.

Theorem 31 (Telescoping series)**.** The series $\sum_{n=1}^{\infty}(b_n - b_{n+1})$ converges if and only if the sequence (b_n) converges, in which case

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - \lim_{n\to\infty} b_n.$$

Theorem 32 (Monotone convergence for series)**.** If $a_n \geq 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if its partial sums are bounded, meaning that there exists $M \geq 0$ such that $\sum_{k=1}^{n} a_k \leq M$ for every $n \in \mathbb{N}$. Otherwise, the series diverges to ∞.

Definition 33 (Cauchy series)**.** A series $\sum_{n=1}^{\infty} a_n$ is Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon \qquad \text{for all } m > n > N.$$

Theorem 34 (Cauchy criterion)**.** A series converges if and only if it is Cauchy.

Theorem 35 (Comparison test)**.** If $|a_n| \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $0 \leq b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Comparison with a geometric series leads to the ratio and root tests for the absolute convergence of series.

Definition 36 (Rearrangements)**.** A series $\sum_{m=1}^{\infty} b_m$ is a rearrangement of the series $\sum_{n=1}^{\infty} a_n$ if there exists a one-to-one, onto map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_m = a_{\sigma(m)}$.

Theorem 37. A series converges absolutely if and only if every rearrangement of the series converges, in which case every rearrangement converges to the same sum.

Theorem 38 (Alternating series test)**.** If (a_n) is a decreasing sequence of positive numbers $a_n \geq 0$ such that $a_n \to 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

4 Topology

Definition 39 (Open sets)**.** A set $A \subset \mathbb{R}$ is open if for every $x \in A$ there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset A$.

Definition 40 (Closed sets)**.** A set $A \subset \mathbb{R}$ is closed if it satisfies either one of the following equivalent conditions: (i) $A^c = \mathbb{R} \setminus A$ is open; (ii) The limit of every convergent sequence in A belongs to A.

Theorem 41 (Properties of open and closed sets)**.** (i) The empty set and the set of real numbers are both open and closed. (ii) Arbitrary unions and finite intersections of open sets are open. (iii) Arbitrary intersections and finite unions of closed sets are closed.

Definition 42 (Limit and isolated points)**.** Let $A \subset \mathbb{R}$. A limit point of A is a real number $x \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists $y \in A \cap (x - \epsilon, x + \epsilon)$ with $y \neq x$. A point $x \in A$ is an isolated point of A if there exists $\epsilon > 0$ such that x is the only point of A in $(x - \epsilon, x + \epsilon)$.

Definition 43 (Closure)**.** Let $A \subset \mathbb{R}$. The closure $\bar{A}$ of A is defined by any of the following equivalent conditions: (i) $\bar{A}$ is the intersection of all closed sets that contain A; (ii) $\bar{A}$ is the set of limits of all convergent sequences in A; (ii) $\bar{A} = A \cup L$ where L is the set of limit points of A.

Definition 44 (Compact sets)**.** A set $K \subset \mathbb{R}$ is compact if it satisfies either one of the following equivalent conditions: (i) Every sequence in K has a convergent subsequence whose limit belongs to K; (ii) Every open cover of K has a finite subcover.

Here, an *open cover* of a set A is a collection of open sets whose union includes A, and a *finite subcover* is a finite collection of open sets from the open cover whose union still includes A.

Theorem 45 (Characterization of compact sets)**.** A set $K \subset \mathbb{R}$ is compact if and only if it is closed and bounded.

Theorem 46 (Nested compact set property)**.** If $K_1 \supset K_2 \supset \cdots \supset K_n \cdots$ is a nested sequence of nonempty, compact sets, then $\cap_{n=1}^{\infty} K_n$ is nonempty.

5 Functions

Definition 47 (Functional limit)**.** Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$. Then $\lim_{x \to c} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in A$ with $0 < |x - c| < \delta$.

We also write $f(x) \to L$ as $x \to c$. One can define left, right, and infinite functional limits in a straightforward way.

Theorem 48 (Sequential characterization of functional limits)**.** Suppose that $f : A \to \mathbb{R}$ and c is a limit point of $A \subset \mathbb{R}$. Then $\lim_{x \to c} f(x) = L$ if and only if for every sequence (x_n) in A with $x_n \neq c$ and $x_n \to c$, one has $f(x_n) \to L$.

Corollary 49 (Divergence criterion)**.** If there exists sequences (x_n), (y_n) in A with $x_n \neq c$, $y_n \neq c$ and $x_n \to c$, $y_n \to c$ such that $f(x_n) \to L$, $f(y_n) \to M$ where $L \neq M$, then $\lim_{x \to c} f(x)$ doesn't exist.

Theorem 50 (Algebraic and order properties)**.** Suppose that $f, g : A \to \mathbb{R}$ and $f(x) \to L$, $g(x) \to M$ as $x \to c$. Then: (i) $kf(x) \to kL$ as $x \to c$ for any $k \in \mathbb{R}$; (ii) $f(x) + g(x) \to L + M$ as $x \to c$; (iii) $f(x)g(x) \to LM$ as $x \to c$; (iv) $f(x)/g(x) \to L/M$ as $x \to c$, provided that $M \neq 0$; (v) if $f(x) \leq g(x)$ for every $x \in A \setminus \{c\}$, then $L \leq M$.

Definition 51 (Continuity)**.** A function $f : A \to \mathbb{R}$ is continuous at $c \in A$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for every $x \in A$ with $|x - c| < \delta$, and f is continuous on A if it is continuous at every $c \in A$.

Theorem 52 (Limit definition of continuity)**.** A function $f : A \to \mathbb{R}$ is continuous at a limit point $c \in A$ if and only if $\lim_{x \to c} f(x) = f(c)$, and f is continuous at every isolated point of A.

Theorem 53 (Sequential characterization of continuity)**.** A function $f : A \to \mathbb{R}$ is continuous at $c \in A$ if and only if for every sequence (x_n) in A with $x_n \to c$, one has $f(x_n) \to f(c)$.

Theorem 54 (Algebraic properties)**.** If $f, g : A \to \mathbb{R}$ are continuous at $c \in A$, then kf, $f+g$, fg are continuous at c, and f/g is continuous at c provided that $g(c) \neq 0$.

A *polynomial function* $p : \mathbb{R} \to \mathbb{R}$ is a function of the form

$$p(x) = a_n x^n + a_{n-1} x_{n-1} + \cdots + a_1 x + a_0,$$

where the coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ are constants. A *rational function* $r : A \to \mathbb{R}$ is the ratio of two polynomial functions $r(x) = p(x)/q(x)$; the domain $A \subset \mathbb{R}$ of r excludes the points where $q(x) = 0$.

Corollary 55. A polynomial function is continuous on $\mathbb{R}$, and a rational function is continuous on its domain.

Theorem 56 (Composition of continuous functions)**.** Suppose that $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$ and $f(A) \subset B$. If f is continuous at $c \in A$ and g is continuous at $f(c)$, then $g \circ f : A \to \mathbb{R}$ is continuous at c.

Definition 57 (Uniform continuity)**.** A function $f : A \to \mathbb{R}$ is uniformly continuous on A if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for every $x, y \in A$ with $|x - y| < \delta$.

Theorem 58 (Criterion for failure of uniform continuity)**.** A function $f : A \to \mathbb{R}$ is not uniformly continuous on A if and only if there exist $\epsilon_0 > 0$ and sequences (x_n), (y_n) in A such that $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$ for every $n \in \mathbb{N}$.

Theorem 59 (Continuous image of compact sets)**.** If $f : K \to \mathbb{R}$ is continuous on a compact set K, then the image $f(K)$ is compact.

Theorem 60 (Extreme value)**.** If $f : K \to \mathbb{R}$ is continuous on a compact set K, then f is bounded and attains its maximum and minimum values.

Theorem 61. If $f : K \to \mathbb{R}$ is continuous on a compact set K, then f is uniformly continuous on K.

Theorem 62 (Intermediate value)**.** If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there exists $a < c < b$ such that $f(c) = L$.

Theorem 63 (Continuous preimage of open sets)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ if and only if the preimage $f^{-1}(V)$ of every open set $V \subset \mathbb{R}$ is open.