Advanced Calculus
Math 127B, Winter 2005
Solutions: Final

1. Define $f_n, g_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \frac{nx^2}{1+n^2x^2}, \qquad g_n(x) = \frac{n^2x}{1+n^2x^2}.$$

Show that the sequences (f_n), (g_n) converge pointwise on $[0,1]$, and determine their pointwise limits. Determine (with proof) whether or not each sequence converges uniformly on $[0,1]$.

Solution.

- As $n \to \infty$, we have $f_n \to 0$ and $g_n \to g$ pointwise, where

$$g(x) = \begin{cases} 1/x & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

- Given $\epsilon > 0$, choose $N = 1/\epsilon$. Then $n > N$ implies that

$$|f_n(x)| = \frac{1}{n}\left(\frac{nx^2}{1/n + nx^2}\right) \leq \frac{1}{n} < \epsilon \qquad \text{for all } x \in [0,1].$$

  Therefore f_n converges uniformly to 0.

- The functions g_n are continuous, and their pointwise limit g is discontinuous. Since the uniform limit of continuous functions is continuous, (g_n) does not converge uniformly.

2. Find all points $x \in \mathbb{R}$ where the following power series converges:

$$\sum_{n=0}^{\infty} \frac{1}{1+n2^n} x^n.$$

Solution.

- According to the ratio test, the radius of convergence R of the power series $\sum a_n x^n$ is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

(provided that this limit exists). Hence the radius of convergence of the given power series is

$$\begin{aligned} R &= \lim_{n\to\infty} \frac{1+(n+1)2^{n+1}}{1+n2^n} \\ &= \lim_{n\to\infty} \frac{1/(n2^n)+(1+1/n)2}{1/(n2^n)+1} \\ &= 2. \end{aligned}$$

- When $x = 2$, the series is

$$\sum_{n=0}^{\infty} \frac{2^n}{1+n2^n} = \sum_{n=0}^{\infty} \frac{1}{n+2^{-n}}.$$

Since

$$\frac{1}{n+2^{-n}} \geq \frac{1}{n+1}$$

this series diverges by comparison with the divergent harmonic series

$$\sum_{n=0}^{\infty} \frac{1}{n+1}.$$

- When $x = -2$, the series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{1+n2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2^{-n}},$$

which converges by the alternating series test, since

$$\frac{1}{n + 2^{-n}} \to 0 \qquad \text{as } n \to \infty$$

and is decreasing in n.

- The power series therefore converges for $-2 \leq x < 2$.

3. (a) Prove that the following series converge on $\mathbb{R}$ to continuous functions:

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}, \qquad g(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}.$$

(b) Prove that g is differentiable on $\mathbb{R}$, and $g' = f$.

Solution.

- (a) Since
$$\left| \frac{\cos nx}{n^2} \right| \leq \frac{1}{n^2}, \qquad \left| \frac{\sin nx}{n^3} \right| \leq \frac{1}{n^3}$$
for all $x \in \mathbb{R}$ and
$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \qquad \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$$
the Weierstrass M-test implies that both series converge uniformly on $\mathbb{R}$. Since the terms in the series are continuous, and the uniform limit of continuous functions is continuous, the sums f, g are continuous.

- (b) Since the uniform convergence of Riemann integrable functions implies convergence of their Riemann integrals, we can integrate the series for f term-by-term over the interval $[0, x]$ (or $[x, 0]$ if $x < 0$) to obtain
$$\begin{aligned} \int_0^x f(t)\, dt &= \sum_{n=1}^{\infty} \int_0^x \frac{\cos nt}{n^2}\, dt \\ &= \sum_{n=1}^{\infty} \frac{\sin nx}{n^3} \\ &= g(x). \end{aligned}$$
Since f is continuous, the fundamental theorem of calculus implies that g is differentiable and $g' = f$.

4. Let $a > 0$. Give a definition of the following improper Riemann integral as a limit of Riemann integrals:

$$\int_2^\infty \frac{1}{x(\log x)^a}\,dx.$$

For what values of a does this integral converge?

Solution.

- We define
$$\int_2^\infty \frac{1}{x(\log x)^a}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\log x)^a}\,dx.$$

- Let
$$I(b) = \int_2^b \frac{1}{x(\log x)^a}\,dx.$$
Making the substitution $u = \log x$, we get
$$I(b) = \int_{\log 2}^{\log b} \frac{1}{u^a}\,du.$$
For $a \neq 1$, we have
$$\begin{aligned} I(b) &= \left[\frac{u^{1-a}}{1-a}\right]_{\log 2}^{\log b} \\ &= \frac{(\log b)^{1-a} - (\log 2)^{1-a}}{1-a}, \end{aligned}$$
which diverges as $b \to \infty$ if $a < 1$. If $a > 1$, then
$$I(b) \to \frac{(\log 2)^{1-a}}{a-1} \qquad \text{as } b \to \infty.$$
If $a = 1$, then
$$\begin{aligned} I(b) &= [\log u]_{\log 2}^{\log b} \\ &= \log(\log b) - \log(\log 2) \\ &\to \infty \qquad \text{as } b \to \infty. \end{aligned}$$

- The improper integral therefore converges when $a > 1$, and then
$$\int_2^\infty \frac{1}{x(\log x)^a}\,dx = \frac{(\log 2)^{1-a}}{a-1}.$$

5. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Is f Riemann integrable on $[0,1]$? Prove your answer.

Solution.

- The function f is not Riemann integrable.
- Suppose that $P = \{t_0, t_1, \ldots, t_n\}$ is any partition of $[0,1]$ (so $t_0 = 0$, $t_n = 1$, and $t_{k-1} < t_k$). Since every interval $[t_{k-1}, t_k]$ contains irrational numbers, we have

  $$m(f, [t_{k-1}, t_k]) = \inf \{f(x) : x \in [t_{k-1}, t_k]\} = 0.$$

  The lower Darboux sum of f is therefore given by

  $$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = 0,$$

  and the lower Darboux integral of f is

  $$L(f) = \sup \{L(f, P) : P \text{ is a partition of } [0,1]\} = 0.$$

- Since the rational numbers are dense in any interval, we have

  $$M(f, [t_{k-1}, t_k]) = \sup \{f(x) : x \in [t_{k-1}, t_k]\} = t_k.$$

  Define $\ell : [0,1] \to \mathbb{R}$ by $\ell(x) = x$. Then

  $$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} t_k (t_k - t_{k-1}) \\ &= U(\ell, P). \end{aligned}$$

  Therefore

  $$U(f) = \inf \{U(f, P) : P \text{ is a partition of } [0,1]\} = U(\ell).$$

Since ℓ is Riemann integrable,

$$U(\ell) = \int_0^1 x \, dx = \frac{1}{2}.$$

So $U(f) = 1/2$. Thus $U(f) > L(f)$, and f is not Riemann integrable.

6. Suppose that

$$F(x) = \begin{cases} -x^2 & \text{for } -1 \le x < 0, \\ x^2 + 2 & \text{for } 0 \le x \le 1. \end{cases}$$

Evaluate the Riemann-Stieltjes integral

$$\int_{-1}^{1} e^{x^2} dF(x).$$

Briefly justify your computations.

Solution.

- We write $F = F_1 + F_2$, where

$$\begin{aligned} F_1(x) &= \begin{cases} 0 & \text{for } -1 \le x < 0, \\ 2 & \text{for } 0 \le x \le 1, \end{cases} \\ F_2(x) &= \begin{cases} -x^2 & \text{for } -1 \le x < 0, \\ x^2 & \text{for } 0 \le x \le 1. \end{cases} \end{aligned}$$

- Using standard properties of the Riemann-Stieltjes integral, and its expression for jump and continuously differentiable integrators, we get

$$\begin{aligned} \int_{-1}^{1} e^{x^2} \, dF(x) &= \int_{-1}^{1} e^{x^2} \, dF_1(x) + \int_{-1}^{1} e^{x^2} \, dF_2(x) \\ &= \int_{-1}^{1} e^{x^2} \, dF_1(x) + \int_{-1}^{0} e^{x^2} \, dF_2(x) + \int_{0}^{1} e^{x^2} \, dF_2(x) \\ &= e^0 \cdot 2 + \int_{-1}^{0} e^{x^2} \, d\left(-x^2\right) + \int_{0}^{1} e^{x^2} \, d\left(x^2\right) \\ &= 2 - \int_{-1}^{0} 2xe^{x^2} \, dx + \int_{0}^{1} 2xe^{x^2} \, dx \\ &= 2 - \left[e^{x^2}\right]_{-1}^{0} + \left[e^{x^2}\right]_{0}^{1} \\ &= 2 - (1 - e) + (e - 1) \\ &= 2e. \end{aligned}$$

7. (a) Find the Taylor series of e^{-x} (at $x = 0$).
(b) Give an expression for the remainder $R_n(x)$ between e^{-x} and its Taylor polynomial of degree $n-1$ involving an intermediate point y between 0 and x.
(c) Prove from your expression in (b) that the Taylor series for e^{-x} converges to e^{-x} for every $x \in \mathbb{R}$. (Don't use general theorems.)

Solution.

- (a) Let $f(x) = e^{-x}$. Then
$$f^{(k)}(x) = (-1)^k e^{-x}.$$
The kth Taylor coefficient of f is
$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{(-1)^k}{k!}.$$
The Taylor series of e^{-x} is therefore
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \dots.$$

- (b) By the Taylor remainder theorem,
$$e^{-x} = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} x^k + R_n(x), \tag{1}$$
where
$$R_n(x) = \frac{(-1)^n e^{-y}}{n!} x^n$$
for some y between 0 and x.

- (c) If $x > 0$, then $0 < y < x$ and $e^{-y} < 1$. Hence
$$|R_n(x)| < \frac{x^n}{n!} \to 0 \qquad \text{as } n \to \infty.$$
(Note that if $c_n = x^n/n!$ then $c_{n+1}/c_n = x/(n+1) < 1/2$ for $n > 2x$, so $c_n \to 0$ as $n \to \infty$ for every $x > 0$.) Taking the limit as $n \to \infty$ in (1), we obtain that
$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k.$$

If $x < 0$, then $e^{-y} < e^{-x}$, and the Taylor series also converges, since

$$|R_n(x)| < e^{-x}\frac{|x|^n}{n!} \to 0 \qquad \text{as } n \to \infty.$$

8. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \left[\sin(1/x) - 2\right] & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

(a) Prove that $f(x)$ has a strict maximum at $x = 0$ (i.e. $f(0) > f(x)$ for all $x \neq 0$).
(b) Prove that f is differentiable on $\mathbb{R}$.
(c) Prove that f is not increasing on the interval $(-\epsilon, 0)$ and f is not decreasing on the interval $(0, \epsilon)$ for any $\epsilon > 0$.

Solution.

- (a) We have $f(0) = 0$. If $x \neq 0$, then since $\sin(1/x) \leq 1$

$$f(x) \leq x^2 \left[1 - 2\right] \leq -x^2 < 0.$$

- (b) The function f is differentiable at any nonzero x since it is a product and composition of differentiable functions. At $x = 0$ the function is differentiable, with $f'(0) = 0$, since

$$\lim_{x \to 0} \left\{ \frac{f(x) - f(0)}{x - 0} \right\} = \lim_{x \to 0} \left\{ x \left[\sin\left(\frac{1}{x}\right) - 2 \right] \right\} = 0.$$

- (c) For $x \neq 0$, we compute using the chain and product rules that

$$f'(x) = -\cos\left(\frac{1}{x}\right) + 2x \left[\sin\left(\frac{1}{x}\right) - 2 \right].$$

If $|x| \leq 1/12$ then

$$\left| 2x \left[\sin\left(\frac{1}{x}\right) - 2 \right] \right| \leq 6|x| < \frac{1}{2},$$

so

$$-\cos\left(\frac{1}{x}\right) - \frac{1}{2} < f'(x) < -\cos\left(\frac{1}{x}\right) + \frac{1}{2}.$$

It follows that $f' < 0$ (hence f is strictly decreasing) in any interval where $\cos(1/x) > 1/2$, and $f' > 0$ (hence f is strictly increasing) in any interval where $\cos(1/x) < -1/2$. Since there exist such intervals arbitrarily close to 0, the function f is not increasing throughout any interval $(-\epsilon, 0)$, nor is it decreasing throughout any interval $(0, \epsilon)$.

- This example shows that a differentiable function may attain a maximum at a point even though it's not increasing on any interval to the left of the point or decreasing on any interval to the right.