Sample Integration Questions
Math 127B. Winter, 2005

1. Give an example of a function $f : [0,1] \to \mathbb{R}$ such that f^2 is Riemann integrable, but f is not.

2. Suppose that $f : [a,b] \to \mathbb{R}$ is a bounded, Riemann integrable function. Define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt.$$

Prove that there exists a constant M such that

$$|F(x) - F(y)| \leq M|x - y| \qquad \text{for all } x, y \in [a,b].$$

Is F necessarily differentiable in (a,b)?

3. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuous. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \int_0^x (x - t)g(t)\,dt.$$

Prove that f satisfies the following equations:

$$f''(x) = g(x), \qquad f(0) = f'(0) = 0.$$

4. Define the improper integral

$$\int_0^\infty \frac{\sin x}{x}\,dx$$

as a limit of proper integrals, and prove that it converges.
Hint. Use integration by parts to show that the proper integrals form a Cauchy sequence.

5. Suppose that

$$F(x) = \begin{cases} x^2 & \text{for } 0 \leq x < 2, \\ x^3 & \text{for } 2 \leq x \leq 3. \end{cases}$$

Evaluate the Riemann-Stieltjes integral

$$\int_0^3 x\,dF(x),$$

briefly justifying your computations.