Math 127C, Spring 2006
Final Exam Solutions

1. Define $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{g} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{aligned} \mathbf{f}(x_1, x_2) &= (\sin x_2 - x_1,\ e^{x_1} - x_2)\,, \\ \mathbf{g}(y_1, y_2) &= \left(y_1 y_2,\ y_1^2 + y_2^2\right)\,. \end{aligned}$$

Use the chain rule to compute the matrix of $(\mathbf{g} \circ \mathbf{f})'(0, 0)$.

Solution.

- By the chain rule,

$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{0}) = \mathbf{g}'(\mathbf{f}(\mathbf{0}))\mathbf{f}'(\mathbf{0}).$$

- The Jacobian matrices of $\mathbf{f}'$ and $\mathbf{g}'$ are

$$[\mathbf{f}'(x_1, x_2)] = \begin{pmatrix} -1 & \cos x_2 \\ e^{x_1} & -1 \end{pmatrix}, \quad [\mathbf{g}'(y_1, y_2)] = \begin{pmatrix} y_2 & y_1 \\ 2y_1 & 2y_2 \end{pmatrix},$$

  and $\mathbf{f}(0, 0) = (0, 1)$.

- Thus,

$$[\mathbf{f}'(0, 0)] = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad [\mathbf{g}'(0, 1)] = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

  and

$$\begin{aligned} [(\mathbf{g} \circ \mathbf{f})'(0, 0)] &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}. \end{aligned}$$

2. Define $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathbf{f}(x, y) = \left(x^2 y + x, 6x + y^2\right).$$

What does the inverse function theorem imply about the existence of a local inverse of $\mathbf{f}$ at: (a) $(x, y) = (1, 1)$; (b) $(x, y) = (-1, 1)$?

Solution.

- The partial derivatives of the components of $\mathbf{f}$ exist and are continuous in $\mathbb{R}^2$, so $\mathbf{f}$ is continuously differentiable in $\mathbb{R}^2$, and the inverse funtion theorem applies at points where the derivative of $\mathbf{f}$ is invertible.

- The Jacobian matrix of $\mathbf{f}'$ is

$$[\mathbf{f}'(x, y)] = \begin{pmatrix} 2xy + 1 & x^2 \\ 6 & 2y \end{pmatrix}.$$

- (a) At $(1, 1)$ we have

$$[\mathbf{f}'(1, 1)] = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}.$$

  The determinant of this matrix is zero, so $\mathbf{f}'(1, 1)$ is singular and there is no conclusion from the inverse function theorem.

- (b) At $(-1, 1)$ we have

$$[\mathbf{f}'(-1, 1)] = \begin{pmatrix} -1 & 1 \\ 6 & 2 \end{pmatrix}.$$

  The determinant of this matrix is non-zero, so $\mathbf{f}'(-1, 1)$ is nonsingular. The inverse function theorem implies that there are open sets $U, V \subset \mathbb{R}^2$ with $(-1, 1) \in U$ and $(0, -5) \in V$ such that $\mathbf{f} : U \to V$ is one-to-one and onto. Moreover, the local inverse $\mathbf{f}^{-1} : V \to U$ is continuously differentiable.

3. Let $I = [0,1] \times [0,1]$, and define $f : I \to \mathbb{R}$ by

$$f(x,y) = y \sin(\pi x y).$$

Why is f Riemann integrable over I? Evaluate $\int_I f \, dxdy$, and justify your calculations.

Solution.

- The function is continuous on I, since it is the product and composition of continuous functions. Therefore it is Riemann integrable on I.

- Since f is continuous, the Riemann integral on I is equal to the iterated integrals by Fubini's theorem. Performing the x-integral first followed by the y-integral, and using the fundamental theorem of calculus, we get

$$\begin{aligned}
\int_I f \, dxdy &= \int_0^1 \left(\int_0^1 y \sin(\pi x y) \, dx \right) dy \\
&= \int_0^1 \left[-\frac{1}{\pi} \cos(\pi x y) \right]_{x=0}^{x=1} dy \\
&= \frac{1}{\pi} \int_0^1 [1 - \cos(\pi y)] \, dy \\
&= \frac{1}{\pi} \left[y - \frac{1}{\pi} \sin(\pi y) \right]_{y=0}^{y=1} \\
&= \frac{1}{\pi}.
\end{aligned}$$

4. Let

$$\omega = f dx + dy, \quad \lambda = g dx + dz$$

be one-forms in $\mathbb{R}^3$, where $f(x, y, z)$ and $g(x, y, z)$ are smooth functions.
(a) Calculate $\omega \wedge \lambda$ and express your answer in standard form.
(b) Calculate $d(\omega \wedge \lambda)$ and express your answer in standard form.

Solution.

- (a) Using the anti-symmetry of the wedge product, we compute that

$$\begin{aligned} \omega \wedge \lambda &= (f dx + dy) \wedge (g dx + dz) \\ &= g dy \wedge dx + f dx \wedge dz + dy \wedge dz \\ &= -g dx \wedge dy + f dx \wedge dz + dy \wedge dz. \end{aligned}$$

- (b) From (a), we compute that

$$\begin{aligned} d(\omega \wedge \lambda) &= d(-g dx \wedge dy + f dx \wedge dz + dy \wedge dz) \\ &= -g_z dz \wedge dx \wedge dy + f_y dy \wedge dx \wedge dz \\ &= -(f_y + g_z)\, dx \wedge dy \wedge dz. \end{aligned}$$

5. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} x^2 \sin(1/y) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Determine, with proof, at what points of $\mathbb{R}^2$ the function f is differentiable.

Solution.

- By the chain, product, and quotient rules, the partial derivatives of f exist and are continuous in the open set $E = \{(x, y) : y \neq 0\}$, so f is continuously differentiable in E.

- If $x \neq 0$ then $\lim_{y \to 0} f(x, y)$ does not exist, so f is not continuous at $(x, 0)$, and therefore f is not differentiable.

- We claim that f is differentiable at $(0,0)$ with $f'(0,0) = 0$. To prove this, note that for $\mathbf{h} = (h,k) \in \mathbb{R}^2$ with $k \neq 0$, we have

$$|f(0+h, 0+k) - f(0,0)| = \left| h^2 \sin \frac{1}{k} - 0 \right| \leq h^2 \leq |\mathbf{h}|^2,$$

while if $k = 0$, we have

$$|f(0+h, 0+k) - f(0,0)| = 0.$$

It follows that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{|f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0})|}{|\mathbf{h}|} = 0,$$

which is what we had to prove.

- Thus, f is differentiable at all points $(x,y) \in \mathbb{R}^2$ except those with $y = 0$ and $x \neq 0$.

6. Let $I = [0,1] \times [0,1]$ and define $f : I \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Prove that f is Riemann integrable on I, and evaluate $\int_I f \, dxdy$.

Solution.

- For $N \in \mathbb{N}$, consider a partition $\mathcal{P}_N$ of I into N^2 equal closed rectangles obtained by partitioning each side $[0,1]$ into N equal intervals.

- The infimum of f on every rectangle is 0, so the lower Riemann sum of f associated with the partition $\mathcal{P}_N$ is

$$\mathcal{L}(f; \mathcal{P}_N) = 0.$$

- The supremum of f on a rectangle in the partition is equal to 0 except for those rectangles that intersect the diagonal $x = y$, where the supremum of f is equal to 1. There are $3N - 2$ such rectangles (N on the diagonal and $2(N-1)$ adjacent to the diagonal). The area of each rectangle is $1/N^2$, so it follows that the upper Riemann sum of f associated with $\mathcal{P}$ is given by

$$\mathcal{U}(f; \mathcal{P}_N) = 1 \cdot (3N - 2) \cdot \frac{1}{N^2}$$

- The right-hand side of this equation tends to 0 as $N \to \infty$. It then follows from standard properties of Riemann sums that

$$0 = \sup_{N \in \mathbb{N}} \mathcal{L}(f; \mathcal{P}_N) \leq \underline{\int\!\!\int_I} f \, dxdy \leq \overline{\int\!\!\int_I} f \, dxdy \leq \inf_{N \in \mathbb{N}} \mathcal{U}(f; \mathcal{P}_N) = 0.$$

Hence

$$\underline{\int\!\!\int_I} f \, dxdy = \overline{\int\!\!\int_I} f \, dxdy = 0,$$

so f is Riemann integrable on I, and

$$\int_I f \, dxdy = 0.$$

7. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the natural numbers, and define

$$d_1, d_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$$

by

$$d_1(n, m) = \left| \frac{1}{n} - \frac{1}{m} \right|, \qquad d_2(n, m) = |n - m|.$$

(a) Prove that d_1, d_2 are metrics on $\mathbb{N}$.

(b) Determine whether or not $\mathbb{N}$ is complete with respect each of the metrics d_1, d_2.

Solution.

- (a) Both d_1, d_2 are symmetric, non-negative, and equal to zero if and only if $n = m$. For d_1, we have

$$\begin{aligned}
d_1(n, m) &= = \left| \frac{1}{n} - \frac{1}{m} \right| \\
&= \left| \frac{1}{n} - \frac{1}{p} + \frac{1}{p} - \frac{1}{m} \right| \\
&\leq \left| \frac{1}{n} - \frac{1}{p} \right| + \left| \frac{1}{p} - \frac{1}{m} \right| \\
&\leq d_1(n, p) + d_2(p, m).
\end{aligned}$$

The triangle inequality for d_2 is immediate.

- (b) The metric space $(\mathbb{N}, d_1)$ is not complete.

- For example, consider the sequence $\{x_k\}$ with $x_k = k$. This is a Cauchy sequence with respect to d_1: Given $\epsilon > 0$, choose $N = 2/\epsilon$. Then $j, k > N$ implies that

$$\begin{aligned} d_1(x_j, x_k) &= \left| \frac{1}{j} - \frac{1}{k} \right| \\ &\leq \frac{1}{j} + \frac{1}{k} \\ &< \frac{2}{N} \\ &< \epsilon. \end{aligned}$$

- If $x \in \mathbb{N}$ is any integer, choose $\epsilon = 1/(2x) > 0$. Then for all $k > 2x$ we have

$$d(x_k, x) = \frac{1}{x} - \frac{1}{k} > \frac{1}{2x} = \epsilon,$$

  It follows that $\{x_k\}$ does not converge with respect to d_1 to any $x \in \mathbb{N}$.

- $(\mathbb{N}, d_2)$ is complete. Any Cauchy sequence $\{x_k\}$ is constant from some point on, since $d_2(x_j, x_k) < 1/2$ implies that $x_j = x_k$. Hence, every Cauchy sequence converges.

8. Define a two-form in $\mathbb{R}^3$ by

$$\omega = \left(x^2 + y^2 + z^2\right) dx \wedge dy.$$

Let

$$I = \left\{(u, v) \in \mathbb{R}^2 : 0 \le u \le \pi/2, \quad 0 \le v \le 2\pi\right\}$$

and define the two-cell $\phi : I \longrightarrow \mathbb{R}^3$ (a half-sphere) by $\phi(u, v) = (x, y, z)$ where

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u.$$

Evaluate

$$\int_\phi \omega.$$

Solution.

- From the definition of the integral of a differential form, we have

$$\int_\phi \omega = \int_I \left(x^2 + y^2 + z^2\right) \frac{\partial(x, y)}{\partial(u, v)}\, dudv.$$

- On the surface, we have

$$\begin{aligned} x^2 + y^2 + z^2 &= \sin^2 u \cos^2 v + \sin^2 u \sin^2 v + \cos^2 u \\ &= 1. \end{aligned}$$

- The Jacobian is

$$\begin{aligned} \frac{\partial(x, y)}{\partial(u, v)} &= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \\ &= \begin{vmatrix} \cos u \cos v & -\sin u \sin v \\ \cos u \sin v & \sin u \cos v \end{vmatrix} \\ &= \cos u \sin u. \end{aligned}$$

- It follows that

$$\int_\phi \omega = \int_I \cos u \sin u\, dudv.$$

- Evaluating this integral by use of Fubini's theorem, we get

$$\begin{aligned}\int_\phi \omega &= \int_0^{\pi/2} \left(\int_0^{2\pi} \cos u \sin u \, dv \right) du \\ &= 2\pi \int_0^{\pi/2} \cos u \sin u \, du \\ &= 2\pi \left[\frac{1}{2} \sin^2 u \right]_0^{\pi/2} \\ &= \pi.\end{aligned}$$