Homework Problems: Set 2
Math 127C, Spring 2006

1. Suppose that X is a linear space of dimension n, and $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ are two bases of X. Prove that there is a unique invertible $n \times n$ matrix $[s_{ij}]$ such that if a vector $\mathbf{x} \in X$ has components $[a_i]$ with respect to E and components $[b_j]$ with respect to F, meaning that

$$\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{e}_i, \qquad \mathbf{x} = \sum_{j=1}^{n} b_j \mathbf{f}_j,$$

then

$$a_i = \sum_{j=1}^{n} s_{ij} b_j.$$

2. Suppose that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function with components $\mathbf{f} = (f_1, f_2, \dots, f_m)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$. Prove that $\mathbf{f}(\mathbf{x}) \to \mathbf{L}$ as $\mathbf{x} \to \mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{L} = (L_1, L_2, \dots, L_m) \in \mathbb{R}^m$, if and only if $f_i(\mathbf{x}) \to L_i$ as $\mathbf{x} \to \mathbf{a}$ for every $1 \le i \le m$.

3. Suppose that $A, B \in L(X, Y)$. Prove that

$$\|A + B\| \le \|A\| + \|B\|.$$

4. Prove that the following two definitions of a closed set $F \subset \mathbb{R}^n$ are equivalent: (a) F^c is open; (b) the limit of every convergent sequence in F belongs to F.

5. Define a function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{f}[(x, y)] = (x + y, xy, x^3 + y^2).$$

Prove that $\mathbf{f}$ is differentiable at $(1, 1)$ and compute $\mathbf{f}'[(1, 1)]$.