Supplementary Homework Problems: Set 4
Math 127C, Spring 2006

1. Define $\mathbf{f} : \mathbb{R} \to \mathbb{R}^2$ by

$$\mathbf{f}(t) = (\cos t, \sin t).$$

(a) Compute $\mathbf{f}'(t)$ and $\|\mathbf{f}'(t)\|$. Determine

$$M = \sup_{\pi < t < 2\pi} \|\mathbf{f}'(t)\|,$$

and verify directly that $\mathbf{f}$ satisfies the inequality

$$|\mathbf{f}(2\pi) - \mathbf{f}(\pi)| \leq M|2\pi - \pi|.$$

(b) Show that there is no $\pi \leq \theta \leq 2\pi$ such that

$$\mathbf{f}(2\pi) - \mathbf{f}(\pi) = \mathbf{f}'(\theta)\,(2\pi - \pi)\,.$$

2. Show that the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$Tx = 1 + \log\left(1 + e^x\right)$$

satisfies

$$|Tx - Ty| < |x - y| \qquad \text{for all } x, y \in \mathbb{R} \text{ with } x \neq y,$$

but T does not have any fixed points. Why doesn't this example contradict the contraction mapping theorem?

3. Define the Fibonacci sequence $\{x_n\}$ by $x_0 = 1$, $x_1 = 1$ and

$$x_{n+1} = x_n + x_{n-1} \qquad n \geq 1.$$

Let $r_n = x_{n+1}/x_n$ be the ratio of successive terms in the Fibonacci sequence. Use the contraction mapping theorem to prove the *astounding* fact, described in *breathtakingly* bad prose in *The Da Vinci Code*, that $r_n \to \phi$ as $n \to \infty$, where

$$\phi = \frac{1 + \sqrt{5}}{2}$$

is the 'Golden Ratio'.