Solutions to Supplementary Homework Problems: Set 5
Math 127C, Spring 2006

1. Let $GL(\mathbb{R}^n)$ be the set of all invertible linear maps $A : \mathbb{R}^n \to \mathbb{R}^n$.
(a) Prove that $GL(\mathbb{R}^n)$ is an open subset of the set $L(\mathbb{R}^n)$ of all linear maps on $\mathbb{R}^n$.
(b) Define $\sigma : GL(\mathbb{R}^n) \to GL(\mathbb{R}^n)$ by $\sigma(A) = A^{-1}$. Prove that σ is continuous. Hint. Use the identity

$$B^{-1} - A^{-1} = B^{-1}\,(A - B)\,A^{-1}.$$

Solution.

- We proved in class that if I is the identity map and $\|A\| < 1$, then $(I - A)$ is invertible and

  $$(I - A)^{-1}\mathbf{x} = \mathbf{x} + A\mathbf{x} + A^2\mathbf{x} + A^3\mathbf{x} + \ldots,$$

  where the series is converges with respect to the norm on $\mathbb{R}^n$. For use in (b), we note that

  $$\begin{aligned} \left|(I - A)^{-1}\mathbf{x}\right| &\leq |\mathbf{x}| + |A\mathbf{x}| + |A^2\mathbf{x}| + |A^3\mathbf{x}| + \ldots \\ &\leq |\mathbf{x}| + \|A\|\,|\mathbf{x}| + \|A^2\|, |\mathbf{x}| + \|A^3\|\,|\mathbf{x}| + \ldots \\ &\leq \big(1 + \|A\| + \|A\|^2 + \|A\|^3 + \ldots\big)\,|\mathbf{x}| \\ &\leq \frac{1}{1 - \|A\|}|\mathbf{x}|. \end{aligned}$$

  Thus,

  $$\left\|(I - A)^{-1}\right\| \leq \frac{1}{1 - \|A\|}.$$

- (a) Suppose that $A \in GL(\mathbb{R}^n)$. If $B \in L(\mathbb{R}^n)$ is such that

  $$\|A - B\| < \frac{1}{\|A^{-1}\|},$$

  then $[I - A^{-1}(A - B)]$ is invertible, since

  $$\|A^{-1}(A - B)\| < 1,$$

and therefore

$$B = A\left[I - A^{-1}(A-B)\right]$$

is invertible, with

$$B^{-1} = \left[I - A^{-1}(A-B)\right]^{-1} A^{-1}.$$

Thus, if $A \in GL(\mathbb{R}^n)$, the open ball centered at A of radius $1/\|A^{-1}\|$ is contained in $GL(\mathbb{R}^n)$, so $GL(\mathbb{R}^n)$ is open.

- (b) It follows from the expression for B^{-1} that if $\|A - B\| < \|A^{-1}\|$ then

  $$\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\,\|A-B\|}.$$

  Hence, if $\|A - B\| < 1/(2\|A^{-1}\|)$, we have

  $$\|B^{-1}\| < 2\|A^{-1}\|.$$

  In that case, we find that

  $$\begin{aligned} \|B^{-1} - A^{-1}\| &= \|B^{-1}\,(A-B)\,A^{-1}\| \\ &\leq \|B^{-1}\|\,\|A^{-1}\|\,\|A-B\| \\ &< 2\|A^{-1}\|^2\|A-B\|, \end{aligned}$$

  which implies that the inversion map is continuous at every $A \in GL(\mathbb{R}^n)$.

- In detail, given $\epsilon > 0$, take

  $$\delta = \min\left(\frac{1}{2\|A^{-1}\|}, \frac{\epsilon}{2\|A^{-1}\|^2}\right).$$

  Then $\|B - A\| < \delta$ implies that $\|\sigma(B) - \sigma(A)\| < \epsilon$.

2. At what values of λ is $(0, \lambda)$ a bifurcation point of the the equation

$$x - \lambda \sin x = 0?$$

Solution.

- The equation is $f(x, \lambda) = 0$, where

  $$f(x, \lambda) = x - \lambda \sin x.$$

  One solution is $x = 0$. By the implicit function theorem, a necessary condition for a bifurcation (meaning that $x = 0$ is not a unique local solution for x as a function of λ) is that $D_x f(0, \lambda) = 0$. Since

  $$D_x f(x, \lambda) = 1 - \lambda \cos x,$$

  this condition implies that $\lambda = 1$.

- By looking at the intersections of the graphs $y = x$ and $y = \lambda \sin x$, one can see that a pitchfork bifurcation occurs at $\lambda = 1$, with two nonzero solutions bifurcating from $(0, 1)$ when $\lambda > 1$. (Other 'limit point' bifurcations occur at nonzero values of x for larger values of λ.)

- We will not give a detailed proof that a pitchfork bifurcation occurs at $(0, 1)$. Note, however, that Taylor expanding $f(x, \lambda)$ about $(0, 1)$, we get

  $$f(x, \lambda) = (\lambda - 1)x - \frac{1}{6}x^3 + \dots,$$

  where the dots denote higher order terms in x and $(\lambda - 1)$. Thus, the nonzero solutions of $f(x, \lambda) = 0$ have the expansion

  $$x = \pm\sqrt{6(\lambda - 1)} + \dots \qquad \text{as } \lambda \to 1^+.$$

- To prove the above result, one can, for example, use the bounds

  $$x - \frac{1}{3!}x^3 \leq \sin x \leq x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5, \qquad x > 0,$$

  which follow from Taylor's theorem with remainder, to show that $f(x, \lambda)$ changes sign in a small interval (say $0 < x < c\sqrt{\lambda - 1}$ for a suitable c) when $\lambda > 1$ and λ is close to 1. It follows from the intermediate value theorem that $f(x, \lambda)$ vanishes at some $x = \bar{x}(\lambda)$ in that interval, which proves that there is a sequence of solutions $\bar{x}(\lambda) > 0$ such that $\bar{x}(\lambda) \to 0$ as $\lambda \to 1^+$.

3. Define $\mathbf{f} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ by

$$\mathbf{f}(\mathbf{x}, \lambda) = (f(x, y, \lambda), g(x, y, \lambda)),$$

where $\mathbf{x} = (x, y)$ and

$$f(x, y, \lambda) = x^3 + \lambda y,$$
$$g(x, y, \lambda) = y^3 - \lambda x.$$

Note that $\mathbf{f}(\mathbf{0}, \lambda) = \mathbf{0}$ for all $\lambda \in \mathbb{R}$. For what value of λ is $D_{\mathbf{x}}\mathbf{f}(\mathbf{0}, \lambda)$ not invertible? Show that a bifurcation does *not* occur at this value of λ.

Solution.

- The Jacobian matrix of $\mathbf{f}$ with respect to $\mathbf{x}$ is

$$[D_{\mathbf{x}}\mathbf{f}(\mathbf{x}, \lambda)] = \begin{bmatrix} 3x^2 & \lambda \\ -\lambda & 3y^2 \end{bmatrix}$$

  Hence,

$$[D_{\mathbf{x}}\mathbf{f}(\mathbf{0}, \lambda)] = \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}.$$

  This is not invertible when $\lambda = 0$.

- If $\mathbf{f}(\mathbf{x}, \lambda) = \mathbf{0}$, then

$$xf(x, y, \lambda) + yg(x, y, \lambda) = 0,$$

  which implies that $x^4 + y^4 = 0$, so $(x, y) = (0, 0)$. Thus, the equation has a unique smooth solution $\mathbf{x}(\lambda) = \mathbf{0}$, and no bifurcation occurs.

- As this example shows, the non-invertibility of the derivative is a necessary, but not a sufficient, condition for a bifurcation to occur. Note that the null-space of $D_{\mathbf{x}}\mathbf{f}(\mathbf{0}, 0)$ is two-dimensional in this example; under very general assumptions, a bifurcation always occurs when the null-space is one-dimensional.