Homework Problems: Set 6
Math 127C, Spring 2006

1. Let $Q = \mathbb{Q} \cap [0, 1]$ denote the rational numbers in the interval $[0, 1]$, and $I = [0, 1] \times [0, 1]$ the unit square in $\mathbb{R}^2$. For $x \in Q$, we write $x = p/q$ in lowest terms (meaning that $0 \le p \le q$ are relatively prime integers), and define the set $S(x) \subset I$ by

$$S(x) = \{(m/q, n/q) : m = 0, 1, 2, \ldots, p, n = 0, 1, 2, \ldots, p\}.$$

We define $S \subset I$ by

$$S = \bigcup_{x \in Q} S(x),$$

and $f : I \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) \in S \\ 1 & \text{if } (x, y) \notin S \end{cases}$$

Show that both iterated integrals

$$\int_0^1 \left(\int_0^1 f(x, y)\, dy \right) dx, \int_0^1 \left(\int_0^1 f(x, y)\, dx \right) dy$$

exist and are equal to 1, but f is not Riemann integrable on I.

2. The convolution of two functions $f, g : \mathbb{R} \to \mathbb{R}$ is the function $f * g : \mathbb{R} \to \mathbb{R}$ defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(x - y) g(y)\, dy,$$

provided that the integral on the right hand side exists for every $x \in \mathbb{R}$.

(a) If f, g are continuous functions with compact support, prove that $f * g$ is also a continuous function with compact support.

(b) We define the L^1-norm of a function $f : \mathbb{R} \to \mathbb{R}$ by

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

If f, g are continuous functions with compact support, prove that

$$\|f * g\|_1 \le \|f\|_1 \|g\|_1.$$

3. If $f : [0,\infty) \to \mathbb{R}$ is a continuous function, we define the improper Riemann integral of f on $[0,\infty)$ by

$$\int_0^\infty f(x)\,dx = \lim_{a\to\infty} \int_0^a f(x)\,dx,$$

assuming that this limit exists.

(a) By the use of two integrations by parts, or otherwise, derive the indefinite integral

$$\int \sin x\, e^{-rx}\,dx = -\left(\frac{\cos x + r\sin x}{1+r^2}\right) e^{-rx} + C,$$

where r is a constant.

(b) Evaluate the integral of the function

$$f(x,y) = \sin x\, e^{-xy}$$

over the rectangle $I = [0,a] \times [0,b]$ in two different ways, and let $b \to \infty$. Deduce that

$$\int_0^a \frac{\sin x}{x}\,dx = \frac{\pi}{2} - \int_0^\infty \frac{(\cos a + y \sin a)\, e^{-ay}}{1+y^2}.$$

(c) Prove that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$