Homework Problems: Set 7
Math 127C, Spring 2006

1. Define the one-form

$$\omega = xdy - ydx$$

in $\mathbb{R}^2$. Let $\gamma : [0, 2\pi] \to \mathbb{R}^2$ be the curve

$$\gamma(\theta) = (\cos\theta, \sin\theta),$$

and $\phi : [0, 1] \times [0, 2\pi] \to \mathbb{R}^2$ the surface

$$\phi(r, \theta) = (r\cos\theta, r\sin\theta).$$

Compute $\int_\gamma \omega$ and $\int_\phi d\omega$ and verify that they are equal.

2. Define the two-form

$$\zeta = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{r^3}$$

in $\mathbb{R}^3 \setminus \{0\}$, where $r = (x^2 + y^2 + z^2)^{1/2}$. Let

$$\phi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$$

be the surface

$$\phi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

(a) Prove that $d\zeta = 0$.
(b) Compute $\int_\phi \zeta$.
(c) Does there exist a one-form λ in $\mathbb{R}^3 \setminus \{0\}$ such that $\zeta = d\lambda$?

3. If ω is a k-form and λ is an m-form, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Show that if ω is a k-form and k is odd then $\omega \wedge \omega = 0$. What can you say if $k \geq 2$ is even?

4. Suppose

$$E = E_x dx + E_y dy + E_z dz, \quad H = H_x dx + H_y dy + H_z dz,$$

are one-forms in $\mathbb{R}^3$ with coordinates (x, y, z), where the subscripts denote different components (not partial derivatives),

$$\begin{aligned} B &= B_x dy \wedge dz - B_y dx \wedge dz + B_z dx \wedge dy, \\ D &= D_x dy \wedge dz - D_y dx \wedge dz + D_z dx \wedge dy, \\ J &= J_x dy \wedge dz - J_y dx \wedge dz + J_z dx \wedge dy \end{aligned}$$

are two-forms, and ρ is a zero-form (function). In $\mathbb{R}^4$ with coordinates (x, y, z, t), we define two-forms

$$F = B + E \wedge dt, \qquad G = D - H \wedge dt$$

and a three-form

$$j = \rho dx \wedge dy \wedge dz - J \wedge dt,$$

(a) Show that the equations

$$dF = 0, \qquad dG = 4\pi j$$

are equivalent to Maxwell's equations (in units in which the speed of light is equal to one):

$$\begin{aligned} \frac{\partial \vec{B}}{\partial t} + \operatorname{curl} \vec{E} &= 0, \\ \operatorname{div} \vec{B} &= 0, \\ -\frac{\partial \vec{D}}{\partial t} + \operatorname{curl} \vec{H} &= 4\pi \vec{J}, \\ \operatorname{div} \vec{D} &= 4\pi\rho. \end{aligned}$$

Here, $\vec{E} = (E_x, E_y, E_z)$ is the electric field, $\vec{B} = (B_x, B_y, B_z)$ is the magnetic induction, $\vec{D} = (D_x, D_y, D_z)$ is the electric displacement, $\vec{H} = (H_x, H_y, H_z)$ is the magnetic field, $\vec{J} = (J_x, J_y, J_z)$ is the current density, and ρ is the charge density.

(b) Deduce that $(\rho, \vec{J})$ satisfy the equation of conservation of charge

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{J} = 0.$$