Midterm 1
Math 127C, Spring 2006
No books, notes or calculators.
Give complete proofs of your answers, using any standard theorems, unless stated explicitly otherwise.

1. Suppose that $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map and $X \subset \mathbb{R}^n$ is a vector space. Prove that

$$A(X) = \{\mathbf{y} \in \mathbb{R}^m : \ \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in X\}$$

is a vector space.

2. Suppose that $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ is the function

$$\mathbf{f}\,(x, y) = \left(x^3 + y, x + y^2, xy\right).$$

(a) What is the derivative of $\mathbf{f}$ at $(1, 2)$? (You should explain your answer carefully, but no proof is required.)

(b) *Using only the definition of the derivative*, prove that $\mathbf{f}$ is differentiable at $(1, 2)$ and that the derivative is what you stated in (a).

3. Consider the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(x, y) = (e^x \cos y, e^x \sin y).$$

(a) Prove that $\mathbf{f}$ is differentiable in $\mathbb{R}^2$ and compute its derivative.

(b) Show that the derivative $\mathbf{f}'(x, y)$ is nonsingular for every $(x, y) \in \mathbb{R}^2$.

(c) What is the range of $\mathbf{f}$? Is $\mathbf{f}$ onto? Is $\mathbf{f}$ one-to-one?

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{xy}{x^4+y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Prove that the partial derivatives of f with respect to x and y exist at every point in $\mathbb{R}^2$. At what points do the directional derivatives of f exist in every direction? Where is f is differentiable?

5. Consider the following special case $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ of the Hénon map defined by

$$\mathbf{f}(x, y) = \big(3 - x^2 - y, x\big) .$$

(a) A point $(a, b) \in \mathbb{R}^2$ is said to be a *fixed point* of $\mathbf{f}$ if $\mathbf{f}(a, b) = (a, b)$. Find all fixed points of $\mathbf{f}$, and verify that $(1, 1)$ is a fixed point.

(b) Compute the Jacobian matrix $[\mathbf{f}'(1, 1)]$ of $\mathbf{f}$ at $(1, 1)$.

(c) Let $\mathbf{f}_2 = \mathbf{f} \circ \mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the composition of $\mathbf{f}$ with itself. Use the chain rule to express the Jacobian matrix $[\mathbf{f}_2'(1, 1)]$ of $\mathbf{f}_2$ at $(1, 1)$ in terms of the Jacobian matrix $[\mathbf{f}'(1, 1)]$, and compute it.

(d) For $n = 1, 2, 3, \ldots$, let $\mathbf{f}_n = \mathbf{f} \circ \mathbf{f} \circ \ldots \circ \mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the n-fold composition of $\mathbf{f}$ with itself (so $\mathbf{f}_{n+1} = \mathbf{f}_n \circ \mathbf{f}$). Let $[A_n] = [\mathbf{f}_n'(1, 1)]$ denote the Jacobian matrix of $\mathbf{f}_n$ at $(1, 1)$. Use the Chain rule to express $[A_{n+1}]$ in terms of $[A_n]$. Using proof by induction, or otherwise, show that

$$[A_n] = (-1)^n \begin{bmatrix} n+1 & n \\ -n & -n+1 \end{bmatrix} .$$