Midterm 2
Math 127C, Spring 2006

No books, notes or calculators.
Prove your answers fully unless stated otherwise.
You can use any theorem given in class,
provided you check its hypotheses.

1. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} = (x_1, x_2, \dots x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$, define

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| + \ldots + |x_n - y_n| .$$

(a) Prove that d is a metric on $\mathbb{R}^n$.

(b) Prove that $\mathbb{R}^n$ is complete with respect to the metric d. (You can assume that $\mathbb{R}$ is complete with respect to the standard metric.)

Solution.

- (a) It is immediate that $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ and $d(\mathbf{x}, \mathbf{y}) \geq 0$. Moreover, $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $x_j = y_j$ for every $1 \leq j \leq n$, meaning that $\mathbf{x} = \mathbf{y}$.

- To prove the triangle inequality, note that if $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n$, then

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &= |x_1 - y_1| + |x_2 - y_2| + \ldots + |x_n - y_n| \\ &= |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2| \\ &\qquad + \ldots + |x_n - z_n + z_n - y_n| \\ &\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| \\ &\qquad + \ldots + |x_n - z_n| + |z_n - y_n| \\ &\leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}). \end{aligned}$$

  This verifies that d is a metric.

- (b) Suppose that $\{\mathbf{x}^{(k)} : k = 1, 2, 3, \ldots\}$, is a Cauchy sequence in $\mathbb{R}^n$ with respect to d. We write

$$\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}).$$

For each $1 \leq j \leq n$, we have

$$|x_j - y_j| \leq d(\mathbf{x}, \mathbf{y})$$

so it follows that $\{x_j^{(k)} : k = 1, 2, 3, \ldots\}$ is a Cauchy sequence in $\mathbb{R}$ for each j. Since $\mathbb{R}$ is complete there exists $x_j \in \mathbb{R}$ such that $x_j^{(k)} \to x_j$ as $k \to \infty$.

- Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then we claim that $\mathbf{x}^{(k)} \to \mathbf{x}$ as $k \to \infty$ with respect to d, which proves that $\mathbb{R}^n$ is complete with respect to d.
- Let $\epsilon > 0$ be given. Since $x_j^{(k)} \to x_j$, there exists K_j such that

$$\left| x_j^{(k)} - x_j \right| < \frac{\epsilon}{n} \qquad \text{for all } k > K_j.$$

Let $K = \max\{K_1, K_2, \ldots, K_n\}$. Then if $k > K$, we have

$$\begin{aligned} d(\mathbf{x}^{(k)}, \mathbf{x}) &= \left| x_1^{(k)} - x_1 \right| + \left| x_2^{(k)} - x_2 \right| + \ldots + \left| x_n^{(k)} - x_n \right| \\ &< \frac{\epsilon}{n} + \frac{\epsilon}{n} + \ldots + \frac{\epsilon}{n} \\ &< \epsilon, \end{aligned}$$

which shows that $\mathbf{x}^{(k)} \to \mathbf{x}$ as $k \to \infty$.

2. (a) Let $I = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$. State Fubini's theorem for the Riemann integral of a function $f : I \to \mathbb{R}$.
(b) Define the function $g : I \to \mathbb{R}$ by $g(x,y) = 0$ if $y \neq 1/2$, and

$$g(x,1/2) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that g is Riemann integrable on I.
(c) For $y \in [0,1]$, define $g^y : [0,1] \to \mathbb{R}$ by $g^y(x) = g(x,y)$. Find the upper and lower Riemann integrals of g^y,

$$\overline{\int_0^1} g^y(x)\,dx, \qquad \underline{\int_0^1} g^y(x)\,dx.$$

Is g^y is Riemann integrable on $[0,1]$ for every $y \in [0,1]$?
(d) Does g satisfy Fubini's theorem for the Riemann integral?

Solution.

- Fubini's theorem states that if $f : I \to \mathbb{R}$ is Riemann integrable on I, then the upper and lower Riemann integrals with respect to x

  $$\overline{\int_0^1} f(x,y)\,dx, \qquad \underline{\int_0^1} f(x,y)\,dx.$$

  are Riemann integrable with repect to y, and

  $$\int_I f\,dxdy = \int_0^1 \left(\overline{\int_0^1} f(x,y)\,dx \right) dy = \int_0^1 \left(\underline{\int_0^1} f(x,y)\,dx \right) dy.$$

  (Of course, the same statement is true with the roles of x and y exchanged.)

- (b) Given any $\epsilon > 0$, consider the partition $\mathcal{P}_\epsilon$ of I into three rectangles,

  $$\begin{aligned} R_1 &= \{(x,y) : 0 \le x \le 1, \quad 0 \le y \le 1/2 - \epsilon\}, \\ R_2 &= \{(x,y) : 0 \le x \le 1, \quad 1/2 - \epsilon \le y \le 1/2 + \epsilon\}, \\ R_3 &= \{(x,y) : 0 \le x \le 1, \quad 1/2 + \epsilon \le y \le 1\}. \end{aligned}$$

- The infimum and supremum of g on R_1, R_3 are 0, while the infimum of g on R_2 is 0 and the supremum of g is 1. The area of R_2 is 2ϵ. It follows that the lower and upper Riemann sums of g with respect to $\mathcal{P}_\epsilon$ are $\mathcal{L}(g, \mathcal{P}_\epsilon) = 0$ and $\mathcal{U}(g, \mathcal{P}_\epsilon) = 2\epsilon$, respectively.

- Since
$$\mathcal{L}(g, \mathcal{P}_\epsilon) \leq \overline{\int_I} g\, dxdy \leq \inf_{\epsilon > 0} \mathcal{U}(g, \mathcal{P}_\epsilon)$$
and
$$\mathcal{L}(g, \mathcal{P}_\epsilon) = 0, \qquad \inf_{\epsilon > 0} \mathcal{U}(g, \mathcal{P}_\epsilon) = 0,$$
we conclude that
$$\overline{\int_I} g\, dxdy = 0.$$

- Since
$$\mathcal{L}(g, \mathcal{P}_\epsilon) \leq \underline{\int_I} g\, dxdy \leq \overline{\int_I} g\, dxdy,$$
we see that
$$\underline{\int_I} g\, dxdy = 0.$$
Hence, the upper and lower Riemann integrals are the same, and g is Riemann integrable on I with
$$\int_I g\, dxdy = 0.$$

- (c) The function g^y is not Riemann integrable on $[0, 1]$ when $y = 1/2$, since
$$g^{1/2}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases},$$
and
$$\underline{\int_0^1} g^{1/2}\, dx = 0, \qquad \overline{\int_0^1} g^{1/2}\, dx = 1.$$

- For $y \neq 1/2$, we have $g^y = 0$, which is Riemann integrable, with
$$\underline{\int_0^1} g^y\, dx = \overline{\int_0^1} g^y\, dx = 0.$$

- Fubini's theorem holds for g (as it must, since g is Riemann integrable on I).
- We can verify this explicitly. The lower x-integral is

  $$\underline{\int_0^1} g^y \, dx = 0 \qquad \text{for all } 0 \le y \le 1,$$

  which is Riemann integrable on $[0, 1]$ with

  $$\int_0^1 \left(\underline{\int_0^1} g^y \, dx \right) dy = 0.$$

  The upper x-integral is

  $$\overline{\int_0^1} g^y \, dx = \left\{ \begin{array}{ll} 1 & \text{if } y = 1/2 \\ 0 & \text{if } y \ne 1/2 \end{array} \right. .$$

  Although discontinuous, this function is Riemann integrable on $[0, 1]$ (use a partition that includes an arbitrarily small interval containing $1/2$), and

  $$\int_0^1 \left(\overline{\int_0^1} g^y \, dx \right) dy = 0.$$

3. For $x, u, v \in \mathbb{R}$, consider the equations

$$\begin{aligned} &u + v^3 - x = 0, \\ &u^2 + 3xv - x^3 + x - 1 = 0. \end{aligned}$$

(a) Verify that: (i) if $x = 1$, then a solution for (u, v) is $(u, v) = (1, 0)$; (ii) if $x = 2$, then a solution for (u, v) is $(u, v) = (1, 1)$.

(b) For each of the solutions (i) and (ii) in (a), say whether or not the implicit function theorem guarantees that there is a unique local solution of the equations for $u = f(x)$, $v = g(x)$. If the implicit function theorem does apply, give a complete and precise statement of what it implies.

Solution.

- (a) Easy to check.
- (b) The equations are of the form $\mathbf{F}(\mathbf{u}, x) = 0$, where $\mathbf{u} = (u, v)$ and $\mathbf{F} : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is defined by $\mathbf{F}(\mathbf{u}, x) = (F(u, v, x), G(u, v, x))$ with
$$F(u, v, x) = u + v^3 - x, \qquad G(u, v, x) = u^2 + 3xv - x^3 + x - 1.$$
- The partial derivatives of these functions exist and are continuous in $\mathbb{R}^2 \times \mathbb{R}$, so $\mathbf{F}$ is continuously differentiable. The implicit function theorem therefore applies at points where $D_{\mathbf{u}}\mathbf{F}$ is invertible.
- We compute that the matrix of the derivative with respect to $\mathbf{u}$ is
$$[D_{\mathbf{u}}\mathbf{F}(u, v, x)] = \begin{bmatrix} 1 & 3v^2 \\ 2u & 3x \end{bmatrix}.$$
- For (i), we get
$$[D_{\mathbf{u}}\mathbf{F}(1, 0, 1)] = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$
This matrix has non-zero determinant, so it is invertible.
- The implicit function theorem implies that there are open sets $V \subset \mathbb{R}$ containing 1 and $U \subset \mathbb{R}^2$ containing $(1, 0)$ such that for every x in V, the equation $\mathbf{F}(\mathbf{u}, x) = 0$ has a unique solution for $\mathbf{u}$ that belongs to U. Moreover, the one-to-one onto function $\mathbf{f} : V \to U$ whose value at $x \in V$ is the unique solution $\mathbf{u} \in U$ is continuously differentiable.

- For (ii), we get
$$[D_{\mathbf{u}}\mathbf{F}(1,1,2)] = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$
This matrix has zero determinant, so the derivative is not invertible, and the implicit function theorem does not lead to any conclusion.

4. Define $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ by $\phi(u, v) = (x, y)$, where

$$x = e^u \cos v, \qquad y = e^u \sin v.$$

(a) Why is ϕ continuously differentiable? Compute the Jacobian

$$J = |\det \phi'|$$

of ϕ, and show that $\phi'(u, v)$ is nonsingular for every $(u, v) \in \mathbb{R}^2$.
(b) Define open sets E, U in $\mathbb{R}^2$ by

$$\begin{aligned} E &= \left\{(u, v) \in \mathbb{R}^2 : 0 < u < 1, \quad 0 < v < \pi/2\right\}, \\ U &= \left\{(x, y) \in \mathbb{R}^2 : x > 0, \quad y > 0, \quad 1 < x^2 + y^2 < e^2\right\}. \end{aligned}$$

Show that $\phi : E \to U$ is one-to-one and onto.
(c) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function with compact support contained in U. Use the change of variables theorem to express

$$\int_{\mathbb{R}^2} f(x, y)\, dxdy$$

as an integral with respect to (u, v).

Solution.

- (a) ϕ is continuously differentiable since the partial derivatives of its component functions exist and are continuous everywhere.

- The Jacobian is

  $$J = \begin{vmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{vmatrix} = e^{2u}.$$

  Since this is nonzero, the derivative is invertible.

- (b) ϕ maps (u, v) to the point with polar coordinates (r, θ) where $r = e^u$ and $\theta = v$. Thus $\phi(E)$ consists of the points with $1 < r < e$ and $0 < \theta < \pi/2$, which is U. If $\phi(u, v) = \phi(u', v')$, then elimination of v and v' implies that $e^{2u} = e^{2u'}$, so $u = u'$; and then $\cos v = \cos v'$ implies that $v = v'$ since $0 < v, v' < \pi/2$. Thus, $\phi : E \to U$ is one-to-one and onto.

- (c) Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(u,v) = (f \circ \phi)(u,v)$ if $(u,v) \in E$ and $g(u,v) = 0$ otherwise. Explicitly,

  $$g(u,v) = \begin{cases} f\left(e^u \cos v, e^u \sin v\right) & \text{if } (u,v) \in E \\ 0 & \text{if } (u,v) \notin E \end{cases}$$

  Then g is a continuous function with compact support contained in E. By the change of variables theorem,

  $$\int_{\mathbb{R}^2} f(x,y)\, dxdy = \int_{\mathbb{R}^2} g(u,v) e^{2u}\, dudv.$$

- Alternatively, defining the closed rectangle $J = \overline{E}$ by

  $$J = \left\{(u,v) \in \mathbb{R}^2 : 0 \le u \le 1, \quad 0 \le v \le \pi/2\right\},$$

  and letting I be any closed rectangle such that $U \subset I$, we have

  $$\int_I f(x,y)\, dxdy = \int_J f\left(e^u \cos v, e^u \sin v\right) e^{2u}\, dudv.$$

- There is one slightly tricky detail: it is not true that

  $$\int_{\mathbb{R}^2} f(x,y)\, dxdy = \int_{\mathbb{R}^2} f\left(e^u \cos v, e^u \sin v\right) e^{2u}\, dudv.$$

  The integral on the right-hand side of this equation is not even well-defined since the integrand does not have compact support (except in the trivial case when $f = 0$).

- The difficulty here comes from the fact that although ϕ is locally one-to-one, it is not globally one-to-one. For any $n \in \mathbb{Z}$, ϕ maps the set

  $$E_n = \left\{(u,v) \in \mathbb{R}^2 : 0 < u < 1, \quad 2\pi n < v < 2\pi n + \pi/2\right\}$$

  one-to-one and onto U. Thus, the fact that f has compact support contained in U does not imply that $f \circ \phi$ has compact support contained in E. Instead, the support of $f \circ \phi$ is contained in the unbounded (non-compact) set

  $$\bigcup_{n=-\infty}^{\infty} E_n.$$