Sample Midterm Questions 2
Math 127C, Spring 2006

1. Show that

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

is a metric on $\mathbb{R}$. Is $\mathbb{R}$ complete with respect to this metric?

Solution.

- The properties that $d(x,y) = d(y,x)$, and $d(x,y) \geq 0$, with $d(x,y) = 0$ if and only if $x = y$, are obvious. The only nontrivial part in the proof that d is a metric is the triangle inequality.

- Define $f : [0, \infty) \to \mathbb{R}$ by

$$f(t) = \frac{t}{1+t}.$$

  Then

$$d(x,y) = f(|x-y|).$$

- The function f is monotonic increasing, since it is differentiable and

$$f'(t) = \frac{1}{(1+t)^2} > 0.$$

  Hence, if $0 \leq u \leq v$, then $f(u) \leq f(v)$.

- Suppose that $s, t \geq 0$. Then

$$\begin{aligned} f(s+t) &= \frac{s+t}{1+s+t} \\ &= \frac{s}{1+s+t} + \frac{t}{1+s+t} \\ &\leq \frac{s}{1+s} + \frac{t}{1+t} \\ &\leq f(s) + f(t). \end{aligned}$$

- Suppose that $x, y, z \in \mathbb{R}$. Let

$$u = |x - y|, \quad v = |x - z| + |z - y|, \quad s = |x - z|, \quad t = |z - y|.$$

Then $u \leq v$, by the triangle inequality on $\mathbb{R}$ with the Euclidean metric, and $v = s+t$. It then follows from the definition of d and the properties of f proved above that

$$\begin{aligned} d(x, y) &= f(u) \\ &\leq f(v) \\ &\leq f(s+t) \\ &\leq f(s) + f(t) \\ &\leq d(x, z) + d(z, y). \end{aligned}$$

Thus, d satisfies the triangle inequality, so it defines a metric on $\mathbb{R}$.

- Note that, with respect to the metric d, the distance between any two points in $\mathbb{R}$ is less than 1.

- The metric space $(\mathbb{R}, d)$ is complete. To prove this, we establish upper and lower bounds of $d(x, y)$ in terms of the Euclidean metric $e(x, y) = |x - y|$, and then deduce the completeness of $(\mathbb{R}, d)$ from the completeness of $(\mathbb{R}, e)$.

- In one direction, we have

$$\begin{aligned} d(x, y) &= \frac{|x - y|}{1 + |x - y|} \\ &\leq |x - y|. \end{aligned}$$

- In the other direction, note that since f is monotone increasing and $f(1/2) = 2/3$, we have

$$d(x, y) \leq 2/3 \text{ if and only if } |x - y| \leq 1/2.$$

Thus, if $d(x, y) \leq 2/3$ then

$$\begin{aligned} d(x, y) &= \frac{|x - y|}{1 + |x - y|} \\ &\geq \frac{|x - y|}{1 + 1/2} \\ &\geq \frac{2}{3}|x - y|. \end{aligned}$$

- Now suppose that $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ with respect to the metric d. Given $\epsilon > 0$, let

$$\epsilon' = \min\left(\frac{2}{3}, \frac{2}{3}\epsilon\right) > 0.$$

Since $\{x_n\}$ is Cauchy, we can choose N such that

$$d\left(x_n, x_m\right) < \epsilon' \qquad \text{for all } n, m > N.$$

Then, for all $n, m > N$, we have $d(x_n, x_m) < 2/3$, and it follows that

$$|x_n - x_m| \leq \frac{3}{2}d(x_n, x_m) < \frac{3}{2}\epsilon' \leq \epsilon.$$

- This proves that $\{x_n\}$ is a Cauchy sequence in $(\mathbb{R}, e)$. Since $(\mathbb{R}, e)$ is complete, there exists $x \in \mathbb{R}$ such that $|x_n - x| \to 0$ as $n \to \infty$.

- Since $d(x_n, x) \leq |x_n - x|$, we see that $d(x_n, x) \to 0$ as $n \to \infty$. Thus, there exists $x \in \mathbb{R}$ such that $x_n \to x$ as $n \to \infty$ with respect to d, so $(\mathbb{R}, d)$ is complete.

2. Does the equation

$$x^5 + y^5 + xy + 4 = 0$$

define an implicit function $x = g(y)$ locally near the point $(x, y) = (-2, 2)$? Explain your answer.

Solution.

- The equation is of the form $f(x, y) = 0$ where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$f(x, y) = x^5 + y^5 + xy + 4.$$

This function is continously differentiable in $\mathbb{R} \times \mathbb{R}$ since its partial derivtives exist and are continuous everywhere.

- The partial derivative $D_x f(x, y) = 5x^4 + y$ is nonsingular at $(x, y) = (-2, 2)$, so the implicit function theorem implies that the equation defines an implicit function $g : J \to I$, where J, I are open sets containing $2, -2$ respectively, and for each $y \in J$, $x = g(y)$ is the unique solution of $f(x, y) = 0$ that lies in I.

3. Suppose that $1/2 \leq a \leq 3/2$. Define a function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(x) = x + \frac{1}{2}\left(a - x^2\right)$$

Find a closed bounded interval $I \subset \mathbb{R}$ containing 1 such that $\phi : I \to I$ is a contraction. If $x_0 \in I$, what do the iterates

$$x_{n+1} = x_n + \frac{1}{2}\left(a - x_n^2\right)$$

converge to as $n \to \infty$?

Solution.

- Check that if $I = [1/2, 3/2]$, then $\phi : I \to I$ maps I into itself.
- Check that $|\phi'(x) \leq 1/2$ for $x \in I$, so that *phi* is a contraction on I.
- Conclude that ϕ has a unique fixed point $\bar{x}$ in I, which must equal $\sqrt{a}$. Hence the iterates x_n converge to $\sqrt{a}$ as $n \to \infty$.

4. Use the change of variables formula to transform

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dxdy$$

into an integral with respect to polar coordinates (r, θ), where

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Deduce the value of

$$\int_0^\infty e^{-x^2}\, dx$$

Justify your steps.

Solution.

- We will just give the formal computation. Note that, given the theorems shown in class, justification is required for the use of integrals over $[0, \infty)$, and in the change of variables formula, because the transformation between polar and cartesian coordinates is not one-to-one at the origin, and the Jacobian vanishes there. To resolve these problems, consider integrals over an annulus $\epsilon^2 \leq x^2 + y^2 \leq a^2$, then let $a \to \infty$ and $\epsilon \to 0$.

- Fubini's theorem implies that

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dxdy = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \left(\int_0^\infty e^{-x^2}\,dx\right)^2.$$

- The Jacobian of the transformation from polar to Cartesian coordinates is

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

- By the change of variables theorem, followed by Fubini's theorem,

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dxdy = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\,rdrd\theta = \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty = \frac{\pi}{4}.$$

- Since these integrals are equal, it follows that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$