

HOMEWORK SOLUTIONS: SET 1  
Math 127C, Spring 2006  
Rudin p.239

1. If  $S = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  and  $\mathbf{x}, \mathbf{y} \in \text{span } S$ , then

$$\mathbf{x} = \sum_{j=1}^n a_j \mathbf{x}_j, \quad \mathbf{y} = \sum_{j=1}^n b_j \mathbf{x}_j$$

for some  $a_j, b_j \in \mathbb{R}$ . Hence

$$\mathbf{x} + \mathbf{y} = \sum_{j=1}^n (a_j + b_j) \mathbf{x}_j \in \text{span } S,$$

and for  $c \in \mathbb{R}$

$$c\mathbf{x} = \sum_{j=1}^n (ca_j) \mathbf{x}_j \in \text{span } S,$$

so  $\text{span } S$  is a vector space.

2. Using the definition of  $BA$  as the composition and the linearity of  $A, B$  we have for  $\mathbf{x}, \mathbf{y} \in X$  and  $c \in \mathbb{R}$  that

$$\begin{aligned} BA(\mathbf{x} + \mathbf{y}) &= B(A(\mathbf{x} + \mathbf{y})) \\ &= B(A\mathbf{x} + A\mathbf{y}) \\ &= B(A\mathbf{x}) + B(A\mathbf{y}) \\ &= BA\mathbf{x} + BA\mathbf{y}, \\ BA(c\mathbf{x}) &= B(A(c\mathbf{x})) \\ &= B(cA\mathbf{x}) \\ &= cB(A\mathbf{x}) \\ &= cBA\mathbf{x}, \end{aligned}$$

which proves that  $BA$  is linear.

Suppose that  $A : X \rightarrow Y$  is an invertible linear map. Then  $A^{-1}\mathbf{y} = \mathbf{x}$  if and only if  $A\mathbf{x} = \mathbf{y}$ . Suppose that  $c \in \mathbb{R}$ . Then  $A^{-1}(c\mathbf{y}) = \mathbf{z}$  implies that

$Az = cy$ . If  $c = 0$ , then  $z = 0$  since  $A$  is one-to-one, so  $A^{-1}(0y) = 0y$ . Otherwise,

$$A\left(\frac{1}{c}z\right) = y,$$

since  $A$  is linear, so

$$\begin{aligned} A^{-1}y &= \frac{1}{c}z \\ &= \frac{1}{c}A^{-1}(cy). \end{aligned}$$

Hence,

$$A^{-1}(cy) = cA^{-1}y.$$

If  $A^{-1}y_1 = x_1$ ,  $A^{-1}y_2 = x_2$ , then  $Ax_1 = y_1$ ,  $Ax_2 = y_2$ . Since  $A$  is linear,

$$\begin{aligned} A(x_1 + x_2) &= Ax_1 + Ax_2 \\ &= y_1 + y_2, \end{aligned}$$

so

$$\begin{aligned} A^{-1}(y_1 + y_2) &= x_1 + x_2 \\ &= A^{-1}y_1 + A^{-1}y_2. \end{aligned}$$

It follows that  $A^{-1}$  is a linear map.

If  $A : X \rightarrow Y$  is invertible, then  $A^{-1} : Y \rightarrow X$  is one-to-one since  $A$  is one-to-one, and  $A^{-1}$  is onto since its range is the domain  $X$  of  $A$ . Hence,  $A^{-1}$  is invertible (with inverse equal to  $A$ ).

**3.** Suppose  $Ax_1 = Ax_2$ . Then the linearity of  $A$  implies that  $A(x_1 - x_2) = 0$ , so  $x_1 - x_2 = 0$ , and  $x_1 = x_2$ . That is,  $A$  is one-to-one.

**4.** Suppose  $A : X \rightarrow Y$  is a linear map. If  $y \in \mathcal{R}(A)$ , there exists  $x \in X$  such that  $Ax = y$ . If  $c \in \mathbb{R}$ , then  $A(cx) = cy$ , so  $cy \in \mathcal{R}(A)$ . Similarly, if  $y_1, y_2 \in \mathcal{R}(A)$ , there exists  $x_1, x_2 \in X$  such that  $Ax_1 = y_1$ ,  $Ax_2 = y_2$ . Since  $A$  is linear,

$$A(x_1 + x_2) = y_1 + y_2,$$

so  $y_1 + y_2 \in \mathcal{R}(A)$ . Hence the range  $\mathcal{R}(A) \subset Y$  is a vector space.

If  $x \in \mathcal{N}(A)$ , then  $Ax = 0$ . If  $c \in \mathbb{R}$ , then

$$A(cx) = cAx = 0,$$

so  $c\mathbf{x} \in \mathcal{N}(A)$ . If  $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(A)$ , then

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0},$$

so  $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(A)$ . Hence the nullspace  $\mathcal{N}(A) \subset X$  is a linear space.