HOMEWORK SOLUTIONS: SET 2
Math 127C, Spring 2006
Rudin p. 239

6. The quotient rule (Theorem 5.3 (c) in Rudin) implies that the partial derivatives of f exist at $(x, y) \neq (0, 0)$, with

$$\begin{aligned} D_1 f(x,y) &= \frac{y\left(y^2 - x^2\right)}{\left(x^2+y^2\right)^2}, \\ D_2 f(x,y) &= \frac{x\left(x^2 - y^2\right)}{\left(x^2+y^2\right)^2}. \end{aligned}$$

Since $f(x, 0) = f(0, y) = 0$ for all $x, y \in \mathbb{R}$, the partial derivatives of f exist at $(0, 0)$, with

$$D_1 f(0,0) = D_2 f(0,0) = 0.$$

Remark. Note that although the partial derivatives of f exist for all $(x, y) \in \mathbb{R}^2$, they are not continuous at $(0, 0)$; for example, if $y \neq 0$ then $D_1 f(0, y) = 1/y$, so $D_1 f$ is unbounded in any neighborhood of the origin.

8. For $\mathbf{h} \in \mathbb{R}^n$, there is an open interval $I \subset \mathbb{R}$ containing the origin such that $\mathbf{x} + t\mathbf{h} \in E$ for $t \in I$. Define $g : I \to \mathbb{R}$ by

$$g(t) = f(\mathbf{x} + t\mathbf{h}).$$

Since f is differentiable at $\mathbf{x}$, g is differentiable at 0 and

$$g'(0) = f'(\mathbf{x})\mathbf{h}.$$

Since f has a local maximum at $\mathbf{x}$, g has a local maximum at 0, and therefore $g'(0) = 0$ (by Theorem 5.8 in Rudin). It follows that $f'(\mathbf{x})\mathbf{h} = 0$ for every $\mathbf{h} \in \mathbb{R}^n$, meaning that $f'(\mathbf{x}) = 0$.

9. Choose $\mathbf{a} \in E$, and define

$$\begin{aligned} U &= \left\{\mathbf{x} \in E : \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})\right\}, \\ V &= \left\{\mathbf{x} \in E : \mathbf{f}(\mathbf{x}) \neq \mathbf{f}(\mathbf{a})\right\}. \end{aligned}$$

Then E is the disjoint union of U and V, and U is nonempty. We claim that U, V are open subsets of E (or $\mathbb{R}^n$). The connectedness of E then implies that V is empty, so $\mathbf{f}$ is constant on E.

The set V is open since $\mathbf{f}$ is continuous (implied by its differentiability) and the inverse image of an open set by a continuous function is open.

To show that U is open, suppose $\mathbf{x} \in U$. Since E is open in $\mathbb{R}^n$, there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset E$, where

$$B_\epsilon(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y} - \mathbf{x}| < \epsilon\}.$$

The open ball $B_\epsilon(\mathbf{x})$ is convex. Since $\mathbf{f}'$ is zero on E, the Corollary to Theorem 9.19 of Rudin implies that $\mathbf{f}$ is constant on $B_\epsilon(\mathbf{x})$, and hence equal to $\mathbf{f}(\mathbf{a})$ (since $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$). It follows that $B_\epsilon(\mathbf{x}) \subset U$, so U is open.

Remark. If the domain E is convex, we can prove this result directly as in the proof of Theorem 9.19 of Rudin. For general open sets E, we have to do a little bit of point set topology. We have used the standard definition that a topological space is *connected* if it is not the disjoint union of two nonempty open sets. Rudin's Definition 2.45 seems to be equivalent to this one, but weird.