SOLUTIONS HOMEWORK PROBLEMS: SET 2
Math 127C, Spring 2006
Supplementary Problems

1. Suppose that X is a linear space of dimension n, and $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ are two bases of X. Prove that there is a unique invertible $n \times n$ matrix $[s_{ij}]$ such that if a vector $\mathbf{x} \in X$ has components $[a_i]$ with respect to E and components $[b_j]$ with respect to F, meaning that

$$\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{e}_i, \qquad \mathbf{x} = \sum_{j=1}^{n} b_j \mathbf{f}_j,$$

then

$$a_i = \sum_{j=1}^{n} s_{ij} b_j.$$

Solution.

- Each vector $\mathbf{f}_j \in F$ has a unique expansion with respect to the basis E. We denote the components by $s_{ij} \in \mathbb{R}$, where

  $$\mathbf{f}_j = \sum_{i=1}^{n} s_{ij} \mathbf{e}_i.$$

  Using this equation in the expansion of $\mathbf{x}$ with respect to F, and exchanging the order of summation, we get

  $$\begin{aligned} \mathbf{x} &= \sum_{j=1}^{n} b_j \left(\sum_{i=1}^{n} s_{ij} \mathbf{e}_i \right) \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} s_{ij} b_j \right) \mathbf{e}_i. \end{aligned}$$

  Since the components of $\mathbf{x}$ with respect to E are unique, it follows that

  $$a_i = \sum_{j=1}^{n} s_{ij} b_j.$$

- The uniqueness of the matrix $[s_{ij}]$ follows from the uniqueness of the expansions of the $\mathbf{f}_j$'s with respect to the basis E, and its invertibility follows by using the same argument with the roles of E and F exchanged (details omitted).

2. Suppose that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function with components $\mathbf{f} = (f_1, f_2, \ldots, f_m)$, where $f_i : \mathbb{R}^n \to \mathbb{R}$. Prove that $\mathbf{f}(\mathbf{x}) \to \mathbf{L}$ as $\mathbf{x} \to \mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{L} = (L_1, L_2, \ldots, L_m) \in \mathbb{R}^m$, if and only if $f_i(\mathbf{x}) \to L_i$ as $\mathbf{x} \to \mathbf{a}$ for every $1 \leq i \leq m$.

Solution.

- For each $1 \leq i \leq m$, we have

  $$|f_i(\mathbf{x}) - L_i| \leq |\mathbf{f}(\mathbf{x}) - \mathbf{L}|.$$

  (Absolute value of a scalar on the left, norm of a vector on the right!) Hence, $\mathbf{f}(\mathbf{x}) \to \mathbf{L}$ implies that $f_i(\mathbf{x}) \to L_i$ for each i.

- Conversely, suppose that $f_i(\mathbf{x}) \to L_i$ as $\mathbf{x} \to \mathbf{a}$ for every $1 \leq i \leq m$. Then, given $\epsilon > 0$ there exists $\delta_i > 0$ such that $0 < |\mathbf{x} - \mathbf{a}| < \delta_i$ implies that

  $$|f_i(\mathbf{x}) - L_i| < \frac{\epsilon}{\sqrt{m}}.$$

  Let

  $$\delta = \min\{\delta_1, \ldots, \delta_m\} > 0.$$

  If $0 < |\mathbf{x} - \mathbf{a}| < \delta$, we have

  $$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{L}| &= \left(\sum_{i=1}^{m} |f_i(\mathbf{x}) - L_i|^2 \right)^{1/2} \\ &< \left(\sum_{i=1}^{m} \frac{\epsilon^2}{m} \right)^{1/2} \\ &< \epsilon. \end{aligned}$$

  Hence, $\mathbf{f}(\mathbf{x}) \to \mathbf{L}$ as $\mathbf{x} \to \mathbf{a}$.

3. Suppose that $A, B \in L(X, Y)$. Prove that

$$\|A + B\| \leq \|A\| + \|B\|.$$

Solution.

- Using the triangle inequality, we get for any $\mathbf{x} \in X$ that

  $$\begin{aligned} |(A+B)\mathbf{x}| &= |A\mathbf{x} + B\mathbf{x}| \\ &\leq |A\mathbf{x}| + |B\mathbf{x}| \,. \end{aligned}$$

  It follows that

  $$\begin{aligned} \|A + B\| &= \sup_{|\mathbf{x}|=1} |(A+B)\mathbf{x}| \\ &\leq \sup_{|\mathbf{x}|=1} \{|A\mathbf{x}| + |B\mathbf{x}|\} \\ &\leq \sup_{|\mathbf{x}|=1} |A\mathbf{x}| + \sup_{|\mathbf{x}|=1} |B\mathbf{x}| \\ &\leq \|A\| + \|B\|. \end{aligned}$$

- Note that, in general, we do not have equality in the step

  $$\sup_{|\mathbf{x}|=1} \{|A\mathbf{x}| + |B\mathbf{x}|\} \leq \sup_{|\mathbf{x}|=1} |A\mathbf{x}| + \sup_{|\mathbf{x}|=1} |B\mathbf{x}| \,.$$

4. Prove that the following two definitions of a closed set $F \subset \mathbb{R}^n$ are equivalent: (a) F^c is open; (b) the limit of every convergent sequence in F belongs to F.

Solution.

- Suppose that F^c is open. If $\mathbf{x} \in F^c$, there exists $\epsilon > 0$ such that the ball $B_\epsilon(\mathbf{x})$ is a subset of F^c and contains no points in F. Thus, no sequence in F can converge to $\mathbf{x}$, and any convergent sequence in F must converge to a point in F.
- Conversely, suppose that F^c is not open. Then there exists a point $\mathbf{x} \in F^c$ such that every ball $B_\epsilon(\mathbf{x})$ with $\epsilon > 0$ contains a point in F. For each $n \in \mathbb{N}$, pick $\mathbf{x}_n \in F \cap B_{1/n}(\mathbf{x})$. Then $\{\mathbf{x}_n\}$ is a convergent sequence in F whose limit $\mathbf{x}$ does not belong to F.

5. Define a function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{f}(x, y) = (x + y, xy, x^3 + y^2).$$

Prove that $\mathbf{f}$ is differentiable at $(1, 1)$ and compute $\mathbf{f}'(1, 1)$.

Solution.

- We define the linear (detailed proof?) map $A : \mathbb{R}^2 \to \mathbb{R}^3$ by

  $$A(h, k) = (h + k, h + k, 3h + 2k).$$

  The map A may be obtained by looking at the linear terms in the Taylor expansion of $\mathbf{f}$ at $(1, 1)$, or by computing the Jacobian matrix of $\mathbf{f}$ at $(1, 1)$.

- We define the 'remainder' function $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ by

  $$\mathbf{r}(\mathbf{h}) = \mathbf{f}(1 + h, 1 + k) - \mathbf{f}(1, 1) - A(h, k),$$

  where $\mathbf{h} = (h, k)$. To prove that $\mathbf{f}$ is differentiable at $(1, 1)$ with derivative $\mathbf{f}'(1, 1) = A$, we need to show that

  $$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{r}(\mathbf{h})}{|\mathbf{h}|} = 0.$$

- Making use of the definitions of $\mathbf{f}$ and A, we compute that

  $$\begin{aligned} \mathbf{r}(\mathbf{h}) &= \big(1 + h + 1 + k, (1 + h)(1 + k), (1 + h)^3 + (1 + k)^2\big) \\ &\qquad - (2, 1, 2) - (h + k, h + k, 3h + 2k) \\ &= \big(0, hk, 3h^2 + h^3 + k^2\big). \end{aligned}$$

  For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we have $|x_i| \le |\mathbf{x}|$ and (from the triangle inequality)

  $$|\mathbf{x}| \le |x_1| + |x_2| + \ldots + |x_n|.$$

  It follows that

  $$\begin{aligned} |\mathbf{r}(\mathbf{h})| &\le |hk| + 3|h|^2 + |h|^3 + |k|^2 \\ &\le 5|\mathbf{h}|^2 + |\mathbf{h}|^3. \end{aligned}$$

  Hence,

  $$\frac{|\mathbf{r}(\mathbf{h})|}{|\mathbf{h}|} \le 5|\mathbf{h}| + |\mathbf{h}|^2 \to 0 \qquad \text{as } \mathbf{h} \to 0,$$

  which proves the result.