Solutions: Problem Set 1
Math 201A, Fall 2006

Problem 1. Give an ϵ-δ proof that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

when $|x| < 1$.

Solution.

- From the formula for the sum of a geometric series, we have

  $$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$

  when $x \neq 1$. Using this result, and assuming that $|x| < 1$, we find that

  $$\left|\sum_{n=0}^{N} x^n - \frac{1}{1-x}\right| \le \frac{|x|^{N+1}}{1-|x|}.$$

  Let $\epsilon > 0$ be given. Since $|x|^n \to 0$ as $n \to \infty$, there exists an $N_\epsilon \in \mathbb{N}$ such that $n > N_\epsilon + 1$ implies that

  $$|x|^n < (1-|x|)\epsilon.$$

  Then $N > N_\epsilon$ implies that

  $$\left|\sum_{n=0}^{N} x^n - \frac{1}{1-x}\right| < \epsilon,$$

  which proves the result.

- Although it could simply be assumed, we prove that if $0 \le x < 1$ then $x^n \to 0$ as $n \to \infty$. The sequence (x^n) is monotone decreasing and bounded below by 0, so the completeness of the reals implies that it approaches a limit ℓ. It follows that

$$\begin{aligned} \ell &= \lim_{n\to\infty} x^n \\ &= \lim_{n\to\infty} x \cdot x^{n-1} \\ &= x \lim_{n\to\infty} x^{n-1} \\ &= x\ell. \end{aligned}$$

Since $x \neq 1$, we must have $\ell = 0$.

Problem 2. If x, y, z are points in a metric space (X, d), show that

$$\begin{aligned} d(x,y) &\geq |d(x,z) - d(y,z)|, \\ d(x,y) + d(z,w) &\geq |d(x,z) - d(y,w)|. \end{aligned}$$

Prove that if $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$.

Solution.

- The triangle inequality

$$d(x,y) + d(y,z) \geq d(x,z)$$

implies that

$$d(x,y) \geq d(x,z) - d(y,z).$$

Exchanging x and y, and using the symmetry of d, we also have

$$d(x,y) \geq d(y,z) - d(x,z).$$

Hence

$$d(x,y) \geq |d(x,z) - d(y,z)|.$$

- The triangle inequality implies that

$$\begin{aligned} d(x,y) &\geq d(x,z) - d(y,z), \\ d(z,w) &\geq d(z,y) - d(w,y). \end{aligned}$$

Adding these equations, we get

$$d(x,y) + d(z,w) \geq d(x,z) - d(w,y).$$

Similarly, we have

$$\begin{aligned} d(y,x) &\geq d(y,w) - d(x,w), \\ d(w,z) &\geq d(w,x) - d(z,x), \end{aligned}$$

and

$$d(y,x) + d(w,z) \geq d(y,w) - d(z,x).$$

Hence

$$d(x,y) + d(z,w) \geq \left| d(x,z) - d(y,w) \right|. \tag{1}$$

- Using (1), we have

  $$0 \leq \left| d\left(x_n, y_n\right) - d(x,y) \right| \leq d(x_n, x) + d\left(y_n, y\right),$$

  which implies that $d\left(x_n, y_n\right) \to d(x,y)$ as $n \to \infty$ if $d\left(x_n, x\right) \to 0$ and $d\left(y_n, y\right) \to 0$.

Problem 3. If (X, d_X) and (Y, d_Y) are metric spaces, show that $d = d_X \times d_Y$ defined by

$$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2),$$

where $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$, is a metric on the Cartesian product $Z = X \times Y$.
If $X = Y = \mathbb{R}$ and $d_X(x,y) = d_Y(x,y) = |x - y|$, describe the set

$$\left\{ z \in \mathbb{R}^2 \mid d(z, 0) < 1 \right\}.$$

Solution.

- We have $d \geq 0$ since $d_X \geq 0$, $d_Y \geq 0$. Moreover, $d(z_1, z_2) = 0$ implies that $d_X(x_1, x_2) = 0$, $d_Y(y_1, y_2) = 0$, so $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$.

- $d(z_2, z_1) = d_X(x_2, x_1) + d_Y(y_2, y_1) = d_X(x_1, x_2) + d_Y(y_1, y_2) = d(z_1, z_2)$.

- If $z_3 = (x_3, y_3)$ then

$$\begin{aligned} d(z_1, z_2) &= d_X(x_1, x_2) + d_Y(y_1, y_2) \\ &\leq d_X(x_1, x_3) + d_X(x_3, x_2) + d_Y(y_1, y_3) + d_Y(y_3, y_2) \\ &\leq d(z_1, z_3) + d(z_3, z_2). \end{aligned}$$

- The set is the interior of the 'diamond' with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

Problem 4. If X is a normed linear space with norm $\|\cdot\|$, define $\rho : X \to \mathbb{R}$ by

$$\rho(x) = \frac{\|x\|}{1 + \|x\|}.$$

(a) Why isn't ρ a norm on X?

(b) Define $r : X \times X \to \mathbb{R}$ by

$$r(x, y) = \rho(x - y).$$

Prove that r is a metric on X.

(c) Define the diameter of X with respect to a metric d by

$$\operatorname{diam}_d(X) = \sup_{x, y \in X} d(x, y).$$

What is the diameter of X with respect to the metric $d(x, y) = \|x - y\|$? What is the diameter of X with respect to the metric $r(x, y) = \rho(x - y)$?

(d) Prove that $\|x_n - x\| \to 0$ as $n \to \infty$ if and only if $r(x_n, x) \to 0$ as $n \to \infty$.

Solution.

- (a) The function ρ is not a norm since it does not satisfy the condition $\rho(\lambda x) = \lambda \rho(x)$ for scalars λ if $x \in X$ is nonzero. (We assume that X is not the trivial normed linear space $\{0\}$.)

- (b) It is clear that $r(x, y) = r(y, x)$, $r(x, y) \geq 0$, and $r(x, y) = 0$ if and only if $x = y$. To prove the triangle inequality, we use the following inequalities, which follow from the proposition below: for $s, t \geq 0$ and $0 \leq t_1 \leq t_2$,

$$\frac{s+t}{1+s+t} \leq \frac{s}{1+s} + \frac{t}{1+t}, \qquad \frac{t_1}{1+t_1} \leq \frac{t_2}{1+t_2}.$$

Since $\|x-y\| \leq \|x\| + \|y\|$, we have

$$\begin{aligned} r(x,y) &= \frac{\|x-y\|}{1+\|x-y\|} \\ &\leq \frac{\|x\|+\|y\|}{1+\|x\|+\|y\|} \\ &\leq \frac{\|x\|}{1+\|x\|} + \frac{\|y\|}{1+\|y\|} \\ &\leq r(x,z) + r(z,y). \end{aligned}$$

Proposition 1 *Suppose that* $f : [0,\infty) \to [0,\infty)$ *is a continuously differentiable function such that* $f(0) = 0$*, and* f' *is non-negative and monotone decreasing. Then for* $s, t \geq 0$

$$0 \leq f(s+t) \leq f(s) + f(t),$$

and for $0 \leq t_1 \leq t_2$

$$0 \leq f(t_1) \leq f(t_2).$$

Proof. If $x, s \geq 0$, then $f'(x+s) \leq f'(x)$, since f' is monotone decreasing. It follows from the fundamental theorem of calculus that

$$\begin{aligned} f(s+t) &= \int_0^{s+t} f'(x)\,dx \\ &= \int_0^s f'(x)\,dx + \int_s^{t+s} f'(x)\,dx \\ &= \int_0^s f'(x)\,dx + \int_0^t f'(x+s)\,dx \\ &\leq \int_0^s f'(x)\,dx + \int_0^t f'(x)\,dx \\ &\leq f(s) + f(t). \end{aligned}$$

Since $f' \geq 0$, the function f is monotone increasing. Hence $0 \leq t_1 \leq t_2$ implies that $f(t_1) \leq f(t_2)$. $\square$

We have

$$r(x, y) = f\left(\|x - y\|\right),$$

where

$$f(t) = \frac{t}{1+t}.$$

Then $f(0) = 0$, and

$$f'(t) = \frac{1}{1+t^2}$$

is non-negative and monotone decreasing, so the inequalities used above follow.

- (c) If $x \in X$ is nonzero, then

$$\sup_{\lambda \in \mathbb{R}} d(\lambda x, 0) = \sup_{\lambda \in \mathbb{R}} \|\lambda x\| = \infty,$$
$$\sup_{\lambda \in \mathbb{R}} r(x, 0) = \sup_{\lambda \in \mathbb{R}} \frac{\|\lambda x\|}{1 + \|\lambda x\|} = 1.$$

  Since $r(x, y) < 1$ for all $x, y \in X$, it follows that

$$\operatorname{diam}_d(X) = \infty, \qquad \operatorname{diam}_r(X) = 1.$$

- (d) Since $0 \leq r(x_n, x) \leq d(x_n, x)$, it follows that $d(x_n, x) \to 0$ implies $r(x_n, x) \to 0$. Conversely if $r(x_n, x) \to 0$, then $f(\|x_n - x\|) \leq 1/2$ for all sufficiently large n. Since f is monotone increasing, it follows that $\|x_n - x\| \leq 1$, and in that case $d(x_n, x) \leq 2r(x_n, x)$. Hence, $r(x_n, x) \to 0$ implies that $d(x_n, x) \to 0$.

Remark. More generally, if (X, d) is any metric space, then (X, d') with metric

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a bounded metric space that has the same topology. There are many other ways to define such a d'; for example

$$d'(x, y) = \max\{d(x, y), 1\}.$$

Problem 5. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the natural numbers, and define

$$d_1, d_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$$

by

$$d_1(n, m) = \left| \frac{1}{n} - \frac{1}{m} \right|, \qquad d_2(n, m) = |n - m| \,.$$

(a) Prove that d_1, d_2 are metrics on $\mathbb{N}$.

(b) Determine whether or not $\mathbb{N}$ is complete with respect each of the metrics d_1, d_2.

Solution.

- (a) It is easy to check that d_1, d_2 are metrics on $\mathbb{N}$.
- (b) The metric space $(\mathbb{N}, d_1)$ is not complete. For example, consider the sequence (x_n) with $x_n = n$. If $\varepsilon > 0$ then $m > n > 1/\varepsilon$ implies that

  $$d_1(x_n, x_m) = \left| \frac{1}{n} - \frac{1}{m} \right| < \frac{1}{n} < \varepsilon,$$

  so the sequence is Cauchy. Suppose that $d(x_n, x) \to 0$ for some $x \in \mathbb{N}$. Then

  $$\frac{1}{x} = \lim_{n\to\infty} \left| \frac{1}{x} - \frac{1}{n} \right| = 0,$$

  which is impossible. Thus, the sequence does not converge.
- The completion of $(\mathbb{N}, d_1)$ can be obtained by adding a point ∞ to $\mathbb{N}$ with $d_1(n, \infty) = 1/n$ for all $n \in \mathbb{N}$. This completion is isometrically isomorphic to the subspace $\{1, 1/2, 1/3, \ldots, 0\}$ of $\mathbb{R}$ equipped with its usual absolute value metric.
- The metric space $(\mathbb{N}, d_2)$ is complete. If (x_n) is a Cauchy sequence, then $d_2(x_n, x_m) < 1$ for all sufficient large n and m, which implies that the terms are the same, and equal to x say. Then the sequence converges to x.
- The metric d_2 gives the discrete topology on $\mathbb{N}$.