Problem Set 2: Solutions
Math 201A: Fall 2016

Problem 1. (a) Prove that a closed subset of a complete metric space is complete. (b) Prove that a closed subset of a compact metric space is compact. (c) Prove that a compact subset of a metric space is closed and bounded.

Solution

- (a) If $F \subset X$ is closed and (x_n) is a Cauchy sequence in F, then (x_n) is Cauchy in X and $x_n \to x$ for some $x \in X$ since X is complete. Then $x \in F$ since F is closed, so F is complete.

- (b) Suppose that $F \subset X$ where F is closed and X is compact. If (x_n) is a sequence in F, then there is a subsequence (x_{n_k}) that converges to $x \in X$ since X is compact. Then $x \in F$ since F is closed, so F is compact. Alternatively, If $\{G_\alpha \subset X : \alpha \in I\}$ is an open cover of F, then $\{G_\alpha : \alpha \in I\} \cup F^c$ is an open cover of X. Since X is compact, there is a finite subcover of X which also covers F, so F is compact.

- (c) Let $K \subset X$ be compact. If (x_n) is a convergent sequence in K with limit $x \in X$, then every subsequence of (x_n) converges to x. Since K is compact, some subsequence of (x_n) converges to a limit in K, so $x \in K$ and K is closed.

- Suppose that K is not bounded, and let $x_1 \in K$. Then for every $r > 0$ there exists $x \in K$ such that $d(x_1, x) \geq r$. Choose a sequence (x_n) in K as follows. Pick $x_2 \in K$ such that $d(x_1, x_2) \geq 1$. Given $\{x_1, x_2, \ldots, x_n\}$, pick $x_{n+1} \in K$ such that

  $$d(x_1, x_{n+1}) \geq 1 + \max_{1 \leq k \leq n} d(x_1, x_k).$$

  By the triangle inequality,

  $$d(x_k, x_{n+1}) \geq d(x_1, x_{n+1}) - d(x_1, x_k) \geq 1 \qquad \text{for every } 1 \leq k \leq n.$$

  It follows that $d(x_m, x_n) \geq 1$ for every $m \neq n$, so (x_n) has no Cauchy subsequences, and therefore no convergent subsequences, so K is not compact.

Problem 2. Let A be a subset of a metric space X with closure $\bar{A}$. Define the interior A° and boundary ∂A of A by

$$A^\circ = \bigcup \{G \subset A : G \text{ is open}\}, \qquad \partial A = \bar{A} \setminus A^\circ.$$

(a) Why is A° open and ∂A closed?

(b) Prove that $X \setminus \bar{A} = (X \setminus A)^\circ$.

(c) Prove that A is closed if and only if $\partial A \subset A$, and A is open if and only if $\partial A \subset A^c$.

(d) If A is open, does it follow that $(\bar{A})^\circ = A$?

Solution

- (a) A union of open sets is open so A° is open, and an intersection of closed sets is closed so $\bar{A}$ and $\partial A = \bar{A} \cap (A^\circ)^c$ are closed.

- (b) Note that $x \in A^\circ$ if and only $B_\epsilon(x) \subset A$ for some $\epsilon > 0$. If $x \in \bar{A}^c$, then $B_\epsilon(x) \subset \bar{A}^c$ for some $\epsilon > 0$ since $\bar{A}^c$ is open. Since $\bar{A} \supset A$, we have $\bar{A}^c \subset A^c$, so $B_\epsilon(x) \subset A^c$, meaning that $x \in (A^c)^\circ$. It follows that $\bar{A}^c \subset (A^c)^\circ$. For the reverse inclusion, note that if $x \in (A^c)^\circ$, then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset A^c$, so x is not the limit of any sequence in A, meaning that $x \in \bar{A}^c$. It follows that $(A^c)^\circ \subset \bar{A}^c$, so $(A^c)^\circ = \bar{A}^c$.

- (c) If A is closed, then $\bar{A} = A$ so $\partial A = A \cap (A^\circ)^c \subset A$. For the converse, note that since $A^\circ \subset A \subset \bar{A}$, we have

  $$\bar{A} = (\bar{A} \cap A^\circ) \cup (\bar{A} \cap (A^\circ)^c) = A^\circ \cup \partial A.$$

  If $\partial A \subset A$, then it follows that $\bar{A} \subset A$, so $A = \bar{A}$, meaning that A is closed.

- (d) This is not true in general. For example, define $A \subset \mathbb{R}^2$ by

  $$A = \left\{(x, y) : x^2 + y^2 < 1\right\} \setminus \{(x, 0) : 0 \leq x < 1\}.$$

  Then A is open, but $(\bar{A})^\circ = \{(x, y) : x^2 + y^2 < 1\} \neq A$.

Problem 3. Let X be a metric space with a dense subset $A \subset X$ such that every Cauchy sequence in A converges in X. Prove that X is complete.

Solution

- Let (x_n) be a Cauchy sequence in X. Since A is dense in X, we can choose a sequence (a_n) in A such that $d(x_n, a_n) \to 0$ as $n \to \infty$. (For example, choose $a_n \in A$ such that $d(x_n, a_n) < 1/n$.)

- Given any $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $d(x_n, a_n) < \epsilon/3$ and $d(x_m, x_n) < \epsilon/3$ for all $m, n > M$, so

$$d(a_m, a_n) \leq d(a_m, x_m) + d(x_m, x_n) + d(x_n, a_n) < \epsilon,$$

  which shows that (a_n) is a Cauchy sequence in A. It follows that (a_n) converges to some $x \in X$.

- Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, a_n) < \epsilon/2$ and $d(a_n, x) < \epsilon/2$ for all $n > N$, so

$$d(x_n, x) \leq d(x_n, a_n) + d(a_n, x) < \epsilon,$$

  which shows that (x_n) converges to x and proves that X is complete.

Problem 4. Let $d : X \times X \to \mathbb{R}$ be the discrete metric on a set X,

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

What are the compact subsets of the metric space (X, d)?

Solution

- A subset of X is compact if and only if it is finite.
- Every finite set is compact. If $F = \{x_1, x_2, \ldots, x_n\} \subset X$ and

  $$\{G_\alpha \subset X : \alpha \in I\}$$

  is an open cover of F, then $x_k \in G_{\alpha_k}$ for some $\alpha_k \in I$, so

  $$\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$$

  is a finite subcover of F. Alternatively, every sequence in F has a constant subsequence, which converges to a point in F.
- Conversely, if $F \subset X$ is infinite and $G_x = \{x\}$, then $\{G_x : x \in F\}$ is an open cover of F with no finite subcover. Alternatively, if F is infinite, then there is a sequence (x_n) in F with $x_m \neq x_n$ for all $m \neq n$, so $d(x_m, x_n) = 1$ for $m \neq n$, and (x_n) has no Cauchy or convergent subsequences.

Remark. A rough heuristic is that compact sets have many properties in common with finite sets. For example, finite sets have the finite intersection property.

Problem 5. Let c_0 be the Banach space of real sequences (x_n) such that $x_n \to 0$ as $n \to \infty$ with the sup-norm $\|(x_n)\| = \sup_{n\in\mathbb{N}} |x_n|$. Is the closed unit ball

$$B = \{(x_n) \in c_0 : \|(x_n)\| \leq 1\}$$

compact?

Solution

- The closed unit ball in c_0 is not compact.
- For example, let

$$\mathbf{e}_k = (\delta_{nk})_{n=1}^{\infty} \qquad \delta_{nk} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

  denote the sequence whose kth term is one and whose other terms are zero. Then $\mathbf{e}_k \in c_0$ since $\lim_{n\to\infty} \delta_{nk} = 0$, and $\mathbf{e}_k$ belongs to the closed unit ball in c_0 since $\|\mathbf{e}_k\| = 1$. However, $\|\mathbf{e}_j - \mathbf{e}_k\| = 1$ for every $j \neq k$, so $(\mathbf{e}_k)_{k=1}^{\infty}$ has no Cauchy or convergent subsequences in c_0.

Remark. A similar argument using the Riesz lemma shows that the closed unit ball in any infinite-dimensional normed space is not compact in the norm topology.

Problem 6. A metric (or topological) space X is disconnected if there are non-empty open sets $U, V \subset X$ such that $X = U \cup V$ and $U \cap V = \emptyset$. A space is connected if it is not disconnected. A space X is totally disconnected if its only non-empty connected subsets are the singleton sets $\{x\}$ with $x \in X$.

(a) Show that the interval $[0, 1]$ is connected (in its standard metric topology).

(b) Show that the set $\mathbb{Q}$ of rational numbers is totally disconnected.

Solution

- (a) Suppose for contradiction that $[0, 1] = U \cup V$ where U, V are nonempty, disjoint open sets in $[0, 1]$. We assume that $0 \in U$ without loss of generality.

- Let $a = \sup\{x \in [0, 1] : [0, x) \subset U\}$. Since $0 \in U$ and U is open, we have $[0, \epsilon) \subset U$ for some $\epsilon > 0$, so $0 < a \leq 1$. If $0 < b < a$, then $[0, b) \subset U$ since, by the definition of the supremum, there exists $b < c < a$ such that $[0, c) \subset U$. (It also follows that $[0, a) = \bigcup_{0<b<a}[0, b) \subset U$, so the supremum is attained and, in fact, $\{x \in [0, 1] : [0, x) \subset U\} = [0, a]$.)

- If $a \in U$, then $(a-\epsilon, a+\epsilon) \subset U$ for some $\epsilon > 0$, but then $[0, a+\epsilon) \subset U$, contradicting the definition of a. On the other hand, if $a \in V$, then $(a-\epsilon, a] \subset V$ for some $0 < \epsilon < a$, but then $[0, b) \not\subset U$ for $a-\epsilon < b \leq a$, also contradicting the definition of a.

- (b) Let $A \subset \mathbb{Q}$ be any subset of the rational numbers with at least two elements. Choose $x, y \in A$ with $x \neq y$. The irrational numbers are dense in $\mathbb{R}$, so there exists $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$. Let

  $$U = (-\infty, z) \cap A, \qquad V = (z, \infty) \cap A.$$

  Then U, V are open sets in the relative topology on A. Moreover, $x \in U$, $y \in V$ so U, V are nonempty, and $U \cap V = \emptyset$, $U \cup V = A$. It follows that $\mathbb{Q}$ is totally disconnected.

Problem 7. Let

$$\mathcal{T} = \{G \subset \mathbb{R} : G^c \text{ is countable or } G^c = \mathbb{R}\}.$$

(a) Show that $\mathcal{T}$ is a topology on $\mathbb{R}$.

(b) Let $I = (0,1)$ with closure $\bar{I} = \bigcap \{F \supset I : F \text{ is closed}\}$ in this topology. Show that $\bar{I} = \mathbb{R}$.

(c) Is there a sequence (x_n) such that $x_n \in I$ and $x_n \to 2$ in this topology?

(d) What part of your proof in Problem 5 of Set 1 fails in this example?

Solution

- (a) The collection of closed sets $\mathcal{C} = \{F \subset \mathbb{R} : F^c \in \mathcal{T}\}$ in this topology is given by $\mathcal{C} = \{F \subset \mathbb{R} : F \text{ is countable or } F = \mathbb{R}\}$.

- The collection $\mathcal{C}$ satisfies the axioms for closed sets in a topological space: (1) $\emptyset, \mathbb{R} \in \mathcal{C}$. (2) The intersection of closed sets is closed, since either every set is $\mathbb{R}$ and the intersection is $\mathbb{R}$, or at least one set is countable and the intersection in countable, since any subset of a countable set is countable. (3) A finite union of closed sets is closed, since a finite (or countable) union of countable sets is countable. It follows that $\mathcal{T}$ is a topology on $\mathbb{R}$ (called the co-countable topology).

- (b) If $F \supset I$ is closed, then F is uncountable, since $(0,1)$ is uncountable, so $F = \mathbb{R}$ and $\bar{I} = \mathbb{R}$.

- (c) Let $(x_n)_{n=1}^{\infty}$ be a sequence in $(0,1)$. Then $U = \mathbb{R} \setminus \{x_n : n \in \mathbb{N}\}$ is an open neighborhood of 2, but $x_n \notin U$ for any $n \in \mathbb{N}$, so (x_n) does not converge to 2. A similar argument applies to any $x \notin (0,1)$, so the sequential closure of I is $\tilde{I} = (0,1)$.

- (d) If X is a topological space, then a neighborhood base of $x \in X$ is a collection $\{U_\alpha : \alpha \in A\}$ of neighborhoods of x such that for every neighborhood U of x there exists $\alpha \in A$ with $U_\alpha \subset U$. Then $x_n \to x$ if and only if for every $\alpha \in A$ there exists $N \in \mathbb{N}$ such that $x_n \in U_\alpha$ for all $n > N$. The proof that the sequential closure is equal to the closure fails for the co-countable topology on $\mathbb{R}$ because $x \in \mathbb{R}$ does not have a countable neighborhood base. On the other hand, if X is a metric space, then $\{B_{1/n}(x) : n \in \mathbb{N}\}$ is a countable neighborhood base of any $x \in X$.

Problem 8. Let S be the set of sequences whose terms are 0 or 1:

$$S = \{(s_k)_{k=1}^{\infty} : s_k = 0 \text{ or } s_k = 1\}.$$

(a) Use a diagonal argument to show that S is uncountable.

(b) Show that S has the same cardinality as the power set $\mathcal{P}(\mathbb{N})$ of the natural numbers.

Solution

- (a) Let $f : \mathbb{N} \to S$. Suppose that $f(n) = \mathbf{s}_n$ where $\mathbf{s}_n = (s_{kn})_{k=1}^{\infty}$, and define $\mathbf{t} = (t_k)_{k=1}^{\infty} \in S$ by

  $$t_k = \begin{cases} 1 & \text{if } s_{kk} = 0, \\ 0 & \text{if } s_{kk} = 1. \end{cases}$$

  Then $\mathbf{t} \neq \mathbf{s}_k$ for every $k \in \mathbb{N}$, since the two sequences have different kth terms. It follows that there is no map from $\mathbb{N}$ onto S, so S is uncountably infinite.

- For $A \subset \mathbb{N}$ define the characteristic function $\chi_A : \mathbb{N} \to \{0, 1\}$ of A by

  $$\chi_A(k) = \begin{cases} 1 & \text{if } k \in A, \\ 0 & \text{if } k \notin A. \end{cases}$$

  Then the map $f : \mathcal{P}(\mathbb{N}) \to S$ defined by $f(A) = (\chi_A(k))_{k=1}^{\infty}$ is one-to-one and onto, so $\mathcal{P}(\mathbb{N})$ and S have the same cardinality.