Midterm: Math 201B
Winter, 2011
Solutions

1.[20 pts.] Say if the following Fourier series represent functions or distributions in $C^\infty(\mathbb{T})$, $C(\mathbb{T})$, $L^2(\mathbb{T})$, or $\mathcal{D}'(\mathbb{T})$:

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sqrt{1+n^4}} e^{inx};$$
$$g(x) \sim \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{1+n^2}} e^{in^2x};$$
$$h(x) \sim \sum_{n=-\infty}^{\infty} e^{-n^4} e^{inx};$$
$$k(x) \sim \sum_{n=-\infty}^{\infty} e^{n^4} e^{inx}.$$

You may use standard theorems proved in class to justify your answers.

Proof:

- First, let's work the details for

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sqrt{1+n^4}} e^{inx}.$$

We can see that the n-th Fourier coefficient is given by

$$\hat{f}(n) = \frac{(-1)^n}{\sqrt{1+n^4}}.$$

From here we see that $|\hat{f}(n)|$ behaves as $\frac{1}{n^2}$ and this implies that $f \in A(\mathbb{T})$, which implies that $\boxed{f \in C(\mathbb{T}).}$

More precisely,

$$\sum_{n=-\infty}^{\infty} \left|\frac{(-1)^n}{\sqrt{1+n^4}}\right| \leq \sum_{n=-\infty}^{\infty} \frac{1}{n^2} < \infty.$$

Note that here we used the p-series test that gave us that indeed the series is convergent.

Further we want to investigate if $f \in L^2(\mathbb{T})$ or $\mathcal{D}'(\mathbb{T})$ or $C^\infty(\mathbb{T})$.

To see if $f \in L^2(\mathbb{T})$, we compute the norm of $\|f\|_{L^2(\mathbb{T})}$, and for this we make use of Perceval's theorem:

$$\sum_{n=-\infty}^{\infty} \left|\frac{(-1)^n}{\sqrt{1+n^4}}\right|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{1+n^4} < \sum_{n=-\infty}^{\infty} \frac{1}{n^4} < \infty.$$

Therefore, we got $\boxed{f \in L^2(\mathbb{T})\ .}$

We check now if $f \in C^\infty(\mathbb{T})$. We easily see that $f' \notin C'(\mathbb{T})$, and therefore $\boxed{f' \notin C^\infty(\mathbb{T}).}$ It remais to check if $f \in \mathcal{D}'(\mathbb{T})$. To check this we can see that the Fourier coefficients of f have slow growth, meaning that $(\exists)$ a non-negative integer k and a constatnt C such that

$$|\hat{f}(n)| \le C(1+n^2)^{\frac{k}{2}} \text{ for all } \ n \in \mathbb{Z}. \tag{1}$$

Indeed, there $(\exists)$ a non-negative integer k and a constatnt C such that

$$\frac{1}{\sqrt{1+n^4}} \le C(1+n^2)^{\frac{k}{2}} \text{ for all } \ n \in \mathbb{Z},$$

which implies $\boxed{f \in \mathcal{D}'(\mathbb{T}).}$

- For

$$g(x) \sim \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{1+n^2}} e^{in^2x}.$$

Observe that using Perceval's theorem we can compute the $L^2(\mathbb{T})$ norm of g:

$$\sum_{n=-\infty}^{\infty} \left|\frac{1}{\sqrt{1+n^2}}\right|^2 = \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} \le \sum_{n=-\infty}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore $\boxed{g(x) \in L^2(\mathbb{T}).}$

Clearly,

$$\sum_{n=-\infty}^{\infty} \left|\frac{1}{\sqrt{1+n^2}}\right| = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{1+n^2}}$$

is divergent. So, $g(x) \notin A(\mathbb{T})$.
Since

$$\sum_{n=-\infty}^{\infty} \frac{|n|^{2s}}{1+n^2} < \infty$$

for $s < \frac{1}{2}$, we have $g \in H^s(\mathbb{T})$ for $s < \frac{1}{2}$. This condition is not sufficient for the Sobolev imbedding theorem to imply anything about the continuity of g.
To check if $g \in \mathcal{D}^{'}(\mathbb{T})$, we can see that the criteria (1) is satisfiedf, and hence $\boxed{g \in \mathcal{D}^{'}(\mathbb{T}).}$
Now, we want to see if $g \in \mathcal{C}^{\infty}(\mathbb{T})$, but since the series

$$\sum_{n=-\infty}^{\infty} |n|^k |\hat{g}(n)|^2$$

is a divergent series for all k's. Hence, $\boxed{g \notin \mathcal{C}^{\infty}(\mathbb{T}).}$

- For

$$h(x) \sim \sum_{n=-\infty}^{\infty} e^{-n^4} e^{inx}.$$

The n-th Fourier coefficient of h is given by $\hat{h}(n) = e^{-n^4}$. But this goes to 0 as $n \to \infty$ faster than any polynomial. So, $\boxed{h \in C^{\infty}(\mathbb{T})}$, which obviously implies that $\boxed{h \in C(\mathbb{T})}$ and $\boxed{h \in L^2(\mathbb{T})}$. Note that here we used Sobolev's Imbedding theorem. From (1), we also get that $\boxed{h \in \mathcal{D}^{'}(\mathbb{T}).}$

- For

$$k(x) \sim \sum_{n=-\infty}^{\infty} e^{n^4} e^{inx}.$$

We check to see if $k(x) \in L^2(\mathbb{T})$, but since the series

$$\sum_{n=-\infty}^{\infty} |e^{n^4}|^2$$

is divergent, we get that $\boxed{k(x) \notin L^2(\mathbb{T}).}$ Also, we can observe that the criteria (1) is not satisfied i.e., k has very fast growth (in fact exponential growth). This implies that $\boxed{k \notin \mathcal{D}^{'}(\mathbb{T}).}$ From the same arguments we use aboved we can conclude that $\boxed{k \notin C^{\infty}(\mathbb{T})}$, which trivially implies that $\boxed{k \notin C(\mathbb{T}).}$

2.[20 pts.] Define an operator $K : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$Kf(x) = \int_0^x \left[f(y) - \tilde{f} \right] dy, \qquad \tilde{f} = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx.$$

(a)**[8 pts.]** Show that K is a bounded linear operator on $L^2(\mathbb{T})$.

Proof:
• The boundnedness of the operator K follows from the following:

$$\begin{aligned} \|K\|_{L^2(\mathbb{T})} &= \int_0^{2\pi} | \int_0^x (f(y) - \tilde{f})\, dy|^2\, dx \\ &\leq \int_0^{2\pi} \left[\int_0^x |f(y) - \tilde{f}|\, dy \right]^2 dx \\ &\leq \int_0^{2\pi} \left[\int_0^{2\pi} |f(y) - \tilde{f}|\, dy \right]^2 dx. \end{aligned}$$

Using Cauchy-Schwarz inequality, we get:

$$\begin{aligned} \|K\|_{L^2(\mathbb{T})} \leq& \int_0^{2\pi} \|1\|^2_{L^2(\mathbb{T})} \|f(y) - \tilde{f}\|^2_{L^2(\mathbb{T})}\, dx \\ =&(2\pi)^3 \|f(y) - \tilde{f}\|^2_{L^2(\mathbb{T})} \\ \leq& C(\|f\|^2_{L^2(\mathbb{T})} + \|\tilde{f}\|^2_{L^2(\mathbb{T})}). \end{aligned}$$

But $\|\tilde{f}\|^2_{L^2(\mathbb{T})}$ can be bounded as follows:

$$\begin{aligned} \|\tilde{f}\|^2_{L^2(\mathbb{T})} =& \int_0^{2\pi} \frac{1}{(2\pi)^2} \left[\int_0^{2\pi} f(x)\, dx \right]^2 dy \\ \leq& \|1\|^2_{L^2(\mathbb{T})} \|f\|^2_{L^2(\mathbb{T})}. \end{aligned}$$

Note that we again used Cauchy-Schwarz inequality. Also, denoting $C_2 := \|1\|^2_{L^2(\mathbb{T})}$, we get:

$$\|K\|_{L^2(\mathbb{T})} \leq C\|f\|^2_{L^2(\mathbb{T})} + C_2\|f\|^2_{L^2(\mathbb{T})} \leq \tilde{C}\|f\|^2_{L^2(\mathbb{T})},$$

where $C + C_2 := \tilde{C}$.
Hence K is bounded.

• The linearity of K follows right away from the linearity of the integral.

(b)[**8 pts.**] What is the kernel of K?

Proof:

$$\text{Ker } \; K = \big\{ f \in L^2(\mathbb{T}) \;|\; Kf = 0 \big\}.$$

So, $f \in \text{Ker}\, K$ implies

$$Kf(x) = 0 \Leftrightarrow \int_0^x \left[f(x) - \tilde{f} \right] \, dy = 0 \text{ for all } x \in [0, \pi] \Leftrightarrow f(y) = \tilde{f} \, \forall \, y \in [0, 2\pi] \,.$$

This says that the kernel of K is formed by L^2 constant functions.

We can also see this result via Fourier series by observing that $\tilde{f} = \hat{f}(0)$. Note that it makes sense to write f's Fourier series since $f \in L^2(\mathbb{T})$.
Therefore, we can conclude that

$$\text{Ker} K = \left\{ f \in L^2(\mathbb{T}) \;|\; f(x) = \hat{f}(0) = \tilde{f} \right\} \Leftrightarrow \text{Ker} K = \big\{ f \in L^2(\mathbb{T}) \;|\; f(x) = \text{constant} \big\} \,.$$

(c)[**4 pts.**] ([**2 pts.**]) What is the range of K?([**1 pts.**]) Can you characterize it items of Sobolev spaces? ([**1 pts.**])Is the range of K closed?

Proof:

$$\text{Ran}\, K = \left\{ g \in L^2(\mathbb{T}) \;|\; g(x) = \int_0^x \left[f(y) - \tilde{f} \right] \, dy \right\} \Leftrightarrow$$
$$\text{Ran}\, K = \left\{ g \in L^2(\mathbb{T}) \;|\; g'(x) = f(x) - \tilde{f} \in L^2(\mathbb{T}) \right\}.$$

This implies that $\text{Ran}\, K$ is a subset of $H^1(\mathbb{T})$. More explicitly:

$$\text{Ran}\, K = \big\{ g \in H^1(\mathbb{T}) \;|\; g(0) = 0 \big\} \,.$$

The range is not closed, since the L^2-limit of differentiable functions need not be differentiable. For example,

$$g_n(x) = \sum_{k=1}^{n} \frac{1}{n\pi} \sin(n\pi x) = \int_0^x \sum_{k=1}^{n} \cos(n\pi y) \, dy \in \text{Ran}\, K$$

and $g_n \to g$ in L^2 as $n \to \infty$, as follows from the Fourier sine expansion of functions in $L^2(0,1)$, where

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{n\pi} \sin(n\pi x) \notin \operatorname{Ran} K.$$

3.[20 pts.] Let $\{x_n\}$, $\{y_n\}$ be sequences in a Hilbert space $\mathcal{H}$.

(a)**[12 pts.]** If $x_n \to x$ (strongly) and $y_n \rightharpoonup y$ (weakly) as $n \to \infty$, prove that

$$\langle x_n, y_n \rangle \to \langle x, y \rangle \qquad \text{as } n \to \infty.$$

Proof: Since $\{x_n\}_n$ is weak convergent to x, then the uniformly boundedness principle implies $\{x_n\}_n$ is bounded. Now look at the difference of $\langle x_n, y_n \rangle$ and $\langle x, y \rangle$, we can see

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n, y_n - y \rangle + \langle x_n - x, y \rangle| \\ &\leq \|x_n\| \|y_n - y\| + |\langle x_n - x, y \rangle| \\ &\leq \left(\sup_n \|x_n\| \right) \|y_n - y\| + |\langle x_n - x, y \rangle| \,. \end{aligned}$$

Since $\{x_n\}_n$ converges to x weakly, and $\{x_n\}_n$ is bounded and $\{y_n\}_n$ converges to y strongly, we find

$$\lim_{n \to \infty} |\langle x_n, y_n \rangle - \langle x, y \rangle| = 0,$$

which proves our assertion.

(b)**[8 pts.]** Prove or give a counter-example: if $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ then

$$\langle x_n, y_n \rangle \to \langle x, y \rangle \qquad \text{as } n \to \infty.$$

Proof: The result stated above is false. Take an orthonormal basis of $\mathcal{H}$; for example $\{e_n\}_{n=1}^{\infty}$, where $e_n = (0, 0, 0, \cdots, 0, 1, 0, \cdots)$ is the vector that has 1 on the n-th position and zero in rest. Using Bessel's inequality, we can prove that $e_n \rightharpoonup 0$. Consider the following two sequences

$$\{x_n\}_{n=1}^{\infty} = \{y_n\}_{n=1}^{\infty} = \{e_n\}_{n=1}^{\infty}$$

Then $\langle x_n, y_n \rangle = \langle e_n, e_n \rangle = 1$.
Hence

$$\langle x_n, y_n \rangle \nrightarrow \langle x, y \rangle \qquad \text{as } n \to \infty.$$

4.[20 pts.] (a)**[10 pts.]**If $f \in L^1(\mathbb{T})$, show that

$$\hat{f}(n) = \frac{1}{4\pi} \int_{\mathbb{T}} [f(x) - f(x + \pi/n)]\, e^{-inx}\, dx.$$

Proof: Computing the LHS we get:

$$\begin{aligned}
\frac{1}{4\pi} \int_{\mathbb{T}} [f(x) - f(x + \pi/n)]\, e^{-inx}\, dx =& \frac{1}{4\pi} \int_{\mathbb{T}} f(x) e^{-inx}\, dx - \frac{1}{4\pi} \int_{\mathbb{T}} f(x + \pi/n) e^{-inx}\, dx \\
=& \frac{1}{2}\hat{f}(n) - \frac{1}{4\pi} \int_{\mathbb{T}} f(x + \pi/n) e^{-in(x+\frac{\pi}{n})} e^{i\pi}\, dx \\
=& \frac{1}{2}\hat{f}(n) - \frac{1}{4\pi} e^{i\pi} \int_{\mathbb{T}} f(x + \pi/n) e^{-in(x+\frac{\pi}{n})}\, dx \\
=& \frac{1}{2}\hat{f}(n) + \frac{1}{4\pi} \int_{\mathbb{T}} f(x + \pi/n) e^{-in(x+\frac{\pi}{n})}\, dx \\
=& \frac{1}{2}\hat{f}(n) + \frac{1}{2} \int_{\mathbb{T}} \hat{f}(n)\, dx \\
=& \hat{f}(n).
\end{aligned}$$

Note that we just did a change of variable to get from the third to last row to the second to last row of the above equality.

(b)**[10 pts.]** Suppose that $f \in C(\mathbb{T})$ is Hölder continuous with exponent α, where $0 < \alpha \leq 1$, meaning that there is a constant $M > 0$ such that

$$|f(x + h) - f(x)| \leq M|h|^{\alpha} \qquad \text{for all } x, h \in \mathbb{T}.$$

Show that there is a constant $C > 0$ such that

$$\left|\hat{f}(n)\right| \leq \frac{C}{|n|^{\alpha}} \qquad \text{for all nonzero integers } n.$$

Proof: Indeed

$$\begin{aligned}
\left|\hat{f}(n)\right| =& \frac{1}{4\pi} \left| \int_{\mathbb{T}} [f(x) - f(x + \pi/n)]\, e^{-inx}\, dx \right| \\
\leq& \frac{1}{4\pi} \int_{\mathbb{T}} |f(x) - f(x + \pi/n)|\, dx \\
\leq& \frac{1}{4\pi} \int_{\mathbb{T}} M \left(\frac{\pi}{n}\right)^{\alpha} dx \\
\leq& \frac{1}{4\pi} (2\pi) M \left(\frac{\pi}{n}\right)^{\alpha} \\
=& C \frac{1}{|n|^{\alpha}}, \text{ where } C = \frac{M\pi^{\alpha}}{2}.
\end{aligned}$$

For extra credit, if you have time.

(c)[**10 pts.**] If $0 < \alpha < 1$, show that the function

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} e^{i2^k x}$$

is Hölder continuous with exponent α and that $\hat{f}(n) = 1/n^\alpha$ for $n = 2^k$ (so the above result is optimal). Hint. You can assume the inequality

$$\left|1 - e^{i\theta}\right| \leq |\theta| \qquad \text{for } \theta \in \mathbb{R}.$$

Proof:

• First, if $\hat{f}(n) = \frac{1}{n^\alpha}$ and $n = 2^k$, we obviously get that

$$\hat{f}(n) = \frac{1}{2^{k\alpha}} = \frac{1}{(2^k)^\alpha} = \frac{1}{|n|^\alpha}.$$

• The last thing we need to prove is that

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} e^{i2^k x}$$

is Hölder continuous with exponent α.

The Hölder condition automatically holds for $|h| \geq 1$, say, with $M = 2\|f\|_\infty$, so (by increasing M if necessary) we just need to prove it for $|h| < 1$.

To estimate the difference $|f(x+h) - f(x)|$ we split it in two parts: one with $k \leq N$ and the other with $k > N$, where we will choose N depending on h in an appropriate way:

$$\begin{aligned}
|f(x+h) - f(x)| &= \left| \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} e^{i2^k(x+h)} - \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} e^{i2^k x} \right| \\
&\leq \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} \left| e^{i2^k(x+h)} - e^{i2^k x} \right| \\
&\leq \sum_{k=0}^{N} \frac{1}{2^{k\alpha}} \left| e^{i2^k h} - 1 \right| + \sum_{k=N+1}^{\infty} \frac{1}{2^{k\alpha}} \left| e^{i2^k h} - 1 \right|.
\end{aligned}$$

We estimate the first part by using the inequality in the hint, which works when $2^k h$ is small, and summing the resulting geometric series (where $\alpha < 1$):

$$\begin{aligned}\sum_{k=0}^{N} \frac{1}{2^{k\alpha}} \left|e^{i2^k h} - 1\right| &\leq \sum_{k=0}^{N} \frac{2^k|h|}{2^{k\alpha}} \\ &\leq |h| \sum_{k=0}^{N} 2^{k(1-\alpha)} \\ &\leq \left(\frac{2^{(N+1)(1-\alpha)} - 1}{2^{1-\alpha} - 1} \right) |h|.\end{aligned}$$

We estimate the second term by using $|e^{i\theta} - 1| \leq 2$, which works when $2^{k\alpha}$ is large:

$$\begin{aligned}\sum_{k=N+1}^{\infty} \frac{1}{2^{k\alpha}} \left|e^{i2^k h} - 1\right| &\leq \sum_{k=N+1}^{\infty} \frac{2}{2^{k\alpha}} \\ &\leq \frac{2}{2^{(N+1)\alpha}} \left(\frac{1}{1 - 1/2^{\alpha}} \right).\end{aligned}$$

It follows that there are constants A, B independent of h and N such that

$$|f(x+h) - f(x)| \leq A2^{N(1-\alpha)}|h| + \frac{B}{2^{N\alpha}}.$$

For any $|h| < 1$, we can choose an integer N such that

$$\frac{1}{|h|} \leq 2^N \leq \frac{2}{|h|}.$$

Using this N in the previous inequality, we find that there is a constant C independent of h such that

$$|f(x+h) - f(x)| \leq C|h|^{\alpha}$$