201B, Winter '11, Professor John Hunter
Homework 2 Solutions

1. If $1 \leq p < \infty$, show the trigonometric polynomials are dense in $L^p(\mathbb{T})$.

Proof. For this problem I am going to give two different approches of proving this problem.

First proof is based on the result that was done in class, meaning that the trigonometric polynomials are dense in $C(\mathbb{T})$. This kind of approch will use $\frac{\varepsilon}{2}$ trick.

Let's begin by considering any function $f(x) \in L^p(\mathbb{T})$. The idea is to approximate it somehow with a trigonometric polynomial, because this is what the problem is asking us to do.
For this let $\epsilon > 0$. Since $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ there is a function $g(x) \in C(\mathbb{T})$ such that

$$\|f - g\|_p < \frac{\epsilon}{2}$$

As mentioned above, using the result proved in class, meaning that the trigonometric polynomials are dense in $C(\mathbb{T})$, then there exists a trigonometric polynomial $p(x) \in C(\mathbb{T})$ such that

$$\|g - p\|_\infty < \frac{\epsilon}{2(2\pi)^{1/p}}.$$

Since for all $x \in \mathbb{T}$,

$$|g(x) - p(x)| \leq \|g - p\|_\infty,$$

then

$$\begin{aligned}
\|g - p\|_p &= \left(\int_{\mathbb{T}} |g(x) - p(x)|^p dx \right)^{1/p} \\
&\leq \left(\int_{\mathbb{T}} \|g - p\|_\infty^p dx \right)^{1/p} \\
&< \left(\int_{\mathbb{T}} \frac{\epsilon^p}{2^p 2\pi} dx \right)^{1/p} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Triangle inequality implies:

$$\|f - p\|_p \leq \|f - g\|_p + \|g - p\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

Hence, trigonometric polynomials are dense in $L^p(\mathbb{T})$.

Second proof. The second proof is using problem 3 from this assignment. Another result we will going to use is:

Proposition 0.1. Assume we have $f \in L^1(\mathbb{T})$ and $g(x) = e^{ixt}$ then the convolution of f with g is going to give us a trigonometric polynomial.

Proof.

$$\begin{aligned} f * g &= \int_{\mathbb{T}} f(y)g(x-y)dy \\ &= \int_{\mathbb{T}} f(y)e^{i(x-y)t}dy \\ &= \int_{\mathbb{T}} f(y)e^{ixt}e^{-iyt}dy \\ &= e^{ixt}\hat{f}(y). \end{aligned}$$

So, indeed we got that $f * g$ is a trigonometric polynomial.

□

Note:

1) The above results holds if we assume a more complex form of the trigonometric polynomial g i.e., if we assume $g(x) = \sum_n a_n e^{ixt}$ where $a_n \in \mathbb{R}$.
2) Also, the result it is still true is we assume that $f \in L^p(\mathbb{T})$. This can be proved in the same way we did the proof of the lemma (0.1).

• Fejer kernel is indeed a trigonometric polynomial, hence when we do the convolution it with any function in $L^1(\mathbb{T})$ (or any function $f \in L^p(\mathbb{T})$)is going to give us a trigonometric polynomial. Please verify that indeed this result is true. You will see that indeed is the same proof for L^p case as we did for L^1 case.

• Also, you can use the class notes to conclude that Fejer kernel is an approximate identity. (A good exercise for you is to try proving this fact by yourself, not using the class notes.)

• Putting together all the information we have found so far, we can apply problem 3 and conclude that for any function $f \in L^p(\mathbb{T})$

$$K_n(x) * f(x) \to f(x), \text{as} \quad n \to \infty,$$

in the $L^p(\mathbb{T})$-norm. In other words any function $f \in L^p(\mathbb{T})$ can be approximated with a trigonometric polynomial. Hence, trigonometric polynomials are dense in $L^p(\mathbb{T})$.

□

2. For fixed $z \in \mathbb{C}$, let $J_n(z)$ denote the nth Fourier coefficient of the function $e^{iz\sin x}$, meaning that

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin x}e^{-inx}dx \qquad \text{for } n \in \mathbb{Z}.$$

(a) What is $J_n(0)$ Show that $J_{-n}(z) = (-1)^n J_n(z)$.

Proof. Note that we have

$$\begin{aligned} J_0(0) &= 1 \\ J_n(0) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-inx}\, dx \\ &= \delta_{0,n}. \end{aligned}$$

Since,

$$\begin{aligned} (-1)^n J_n(z) &= \frac{e^{i\pi n}}{2\pi}\int_0^{2\pi} e^{iz\sin x}e^{-inx}\, dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin x}e^{in(\pi-x)}\, dx. \end{aligned}$$

then we can do the change of variables $y = \pi - x$, and obtain

$$-\frac{1}{2\pi}\int_{\pi}^{-\pi} e^{iz\sin(\pi-y)}e^{iny}\, dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iz\sin y}e^{iny}\, dy = J_{-n}(z).$$

□

(b) Derive the recurrence relations

$$\frac{2n}{z}J_n(z) = J_{n-1}(z) + J_{n+1}(z), \qquad 2J_n'(z) = J_{n-1}(z) - J_{n+1}(z),$$

where the prime denotes a derivative with respect to z.

Proof. This is mostly based on integration's techniques. Integration by parts gives us:

$$\begin{aligned}
J_{n-1}(z) + J_{n+1}(z) &= \frac{1}{2\pi}\int_o^{2\pi} e^{iz\sin x}e^{-inx}(e^{ix}+e^{-ix})\,dx \\
&= \frac{2}{2\pi}\int_0^{2\pi} e^{iz\sin x}\cos(x)e^{-inx}\,dx \\
&= \frac{2}{2\pi}\left[\frac{e^{-inx}e^{iz\sin x}}{iz}\Big|_0^{2\pi} - \int_0^{2\pi}\frac{e^{iz\sin x}}{iz}\left(-ine^{-inx}\,dx\right)\right] \\
&= \frac{2n}{z}\frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin x}e^{-inx}\,dx \\
&= \frac{2n}{z}J_n(z).
\end{aligned}$$

For the second relation we have:

$$\begin{aligned}
2J_n'(z) &= \frac{1}{\pi}\frac{d}{dz}\left(\int_0^{2\pi} e^{iz\sin x}e^{-inx}\,dx\right) \\
&= \frac{1}{\pi}\int_0^{2\pi} e^{iz\sin x}e^{-inx}i\sin nx\,dx \\
&= \frac{1}{\pi}\int_0^{2\pi} e^{iz\sin x}e^{-inx}i\frac{e^{ix}-e^{-ix}}{2i}\,dx \\
&= \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin x}e^{-i(n-1)x} - e^{iz\sin x}e^{-i(n+1)x}\,dx \\
&= J_{n-1}(z) - J_{n+1}(z).
\end{aligned}$$

□

(c) Deduce from (b) that $J_n(z)$ is a solution of Bessel's equation

$$z^2J_n'' + zJ_n' + (z^2-n^2)J_n = 0.$$

Proof. We have

$$\begin{aligned}
zJ_n' &= zJ_{n-1} - nJ_n \\
zJ_{n+1} &= nJ_n - zJ_n'
\end{aligned}$$

Differentiating both sides of the first equation from the above system and moving everything on the right side of the equation we see that

$$zJ_n'' + (n+1)J_n' - J_{n-1} - zJ_{n-1}' = 0.$$

Multiplying both sides by z we get:

$$z^2 J_n'' + (n+1)zJ_n' - zJ_{n-1} - z^2 J_{n-1}' = 0$$

Also, by moving everything in the remaining equation to the left side of the equation and multiplying both sides by n we get

$$nzJ_n' - nzJ_{n-1} + n^2 J_n = 0$$

Therefore

$$\begin{aligned} z^2 J_n'' + (n+1)zJ_n' - zJ_{n-1} - z^2 J_{n-1}' - (nzJ_n' - nzJ_{n-1} + n^2 J_n) = \\ = z^2 J_n'' + zJ_n' + z(n-1)J_{n-1} - z^2 J_{n-1}' - n^2 J_n \\ = z^2 J_n'' + zJ_n' + z[(n-1)J_{n-1} - zJ_{n-1}'] - n^2 J_n \\ = 0 \end{aligned}$$

Hence,

$$zJ_n = (n-1)J_{n-1} - zJ_{n-1}'.$$

Substituting this into the previous equation we get

$$z^2 J_n'' + zJ_n' + (z^2 - n^2)J_n = 0$$

□

3. A family of (not necessarily positive) functions $\{\phi_n \in] L^1(\mathbb{T}) : n \in \mathbb{N}\}$ is an approximate identity if

$$\int \phi_n dx = 1 \quad \text{for every } n \in \mathbb{N};$$

$$\int |\phi_n| dx \leq M \quad \text{for some constant } M \text{ and all } n \in \mathbb{N};$$

$$\lim_{n\to\infty} \int_{\delta<|x|<\pi} |\phi_n| dx = 0 \quad \text{for every } \delta > 0.$$

If $f \in L^1(\mathbb{T})$, show that $\phi_n * f \to f$ in $L^1(\mathbb{T})$ as $n \to \infty$.

Proof. The problem is asking to prove that if $f \in L^1(\mathbb{T})$, show that $\phi_n * f \to f$ in $L^1(\mathbb{T})$ as $n \to \infty$, which is equivalent with proving that

$$\|\phi_n * f - f\|_{L^1(\mathbb{T})} \to 0 \text{ as } n \to \infty.$$

The idea is to first prove this result for a continuous function f and then using the desity of the continuous functions in the $L^1(\mathbb{T})$ space for $p \geq 1$ to conclude the desired result.

We will choose a sequence of continuous functions $\{g_n\}_n$ that approximates f in the L^1-norm. Then we will prove that $\|\phi_n * g_n - \phi_n * f\|_{L^1(\mathbb{T})}$ is "small" and at the same time we have that $\|\phi_n * g_n - g_n\|_{L^1(\mathbb{T})}$ is also "small". By small I mean a "quantity" less than $\frac{\varepsilon}{3}$.

Let $\varepsilon > 0$ be given. Since $C(\mathbb{T})$ is dense in $L^1(\mathbb{T})$, then we can find a sequence of functions $\{g_n\}_n$ in $C(\mathbb{T})$ such that

$$\|f - g_n\|_{L^1(\mathbb{T})} < \frac{\varepsilon}{3M},$$

where M is just a positive constant. The reason we picked $\frac{\varepsilon}{3M}$ is just something that usually is done just because at the end of the proof we what to get a nice looking inequality.

Using the first property of the approximate identity, we have the following:

$$\begin{aligned}
\|\phi_n * g_n - \phi_n * f\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} \left| \int_{\mathbb{T}} \phi_n(x-y) g_n(y)\, dy - \int_{\mathbb{T}} \phi_n(x-y) f(y)\, dy \right|\, dx \\
&= \int_{\mathbb{T}} \left| \int_{\mathbb{T}} \phi_n(x-y) \left[g_n(y) - f(y)\right]\, dy \right|\, dx \\
&\leq \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |\phi_n(x-y)||g_n(y) - f(y)|\, dy \right) dx.
\end{aligned}$$

Changing the order of integration (here you need to be careful and see that Fubini's theorem really applies!)

$$\begin{aligned}
\|\phi_n * g_n - \phi_n * f\|_{L^1(\mathbb{T})} &\leq \int_{\mathbb{T}} \left(\int_{\mathbb{T}} |\phi_n(x-y)||g_n(y) - f(y)|\, dy \right) dx \\
&= \left(\int_{\mathbb{T}} |\phi_n(x-y)|\, dx \right) \left(\int_{\mathbb{T}} |g_n(y) - f(y)|\, dy \right) \\
&\leq M \frac{\varepsilon}{3M} \\
&= \frac{\varepsilon}{3}
\end{aligned}$$

At this point we still have to prove that

$$\|\phi_n * g_n - g_n\|_{L^1(\mathbb{T})}$$

is a "small quantity" and afterward using the $\frac{\varepsilon}{3}$ trick, we are done.

Therefore, using again the properties of the approximate identity ϕ_n, we get

$$\begin{aligned}
\|\phi_n * g_n - g_n\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} \mid \int_{\mathbb{T}} \phi_n(x-y) g_n(y)\, dy - g_n(x) \mid \, dx \\
&= \int_{\mathbb{T}} \mid \int_{\mathbb{T}} \phi_n(x-y) g_n(y)\, dy - \int_{\mathbb{T}} \phi_n(x-y) g_n(x)\, dy \mid \, dx \\
&= \int_{\mathbb{T}} \mid \int_{\mathbb{T}} \phi_n(x-y) \left[g_n(y) - g_n(x)\right] \, dy \mid \, dx \\
&\leq \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \mid \phi_n(x-y) \mid\mid g_n(y) - g_n(x) \mid \, dy \right) dx.
\end{aligned}$$

Now, we need to handle the inner integral. The way to do it, is to split the integral in a "wise" way. You will see right away what I mean by wise. An important thing that we didn't use yet, is the continuity of the g_n's.

Hence, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ will imply that

$$|g_n(x) - g_n(y)| < \frac{\varepsilon}{\heartsuit M}.$$

We will figure out later what $\heartsuit$ is. This usually is "fixed" at the end of the proof. In our case it will turn out to be $\heartsuit = 12\pi$.

So, the g_n's are continuous functions on the $\mathbb{T}$, which is a compact set, therefore we have that g_n is a uniform continuous function for every $n \in \mathbb{N}$.

Hence there exists a constant $A > 0$ such that

$$\sup_{x,y \in \mathbb{T}} |g_n(x) - g_n(y)| < A.$$

Then the following holds:

$$
\begin{aligned}
\|\phi_n * g_n - g_n\|_{L^1(\mathbb{T})} &= \int_{\mathbb{T}} \mid \int_{\mathbb{T}} \phi_n(x-y) g_n(y)\,dy - g_n(x) \mid \,dx \\
&\leq \int_{\mathbb{T}} \Big(\int_{|x-y|<\delta} \mid \phi_n(x-y) \mid\mid g_n(y) - g_n(x) \mid \,dy \\
&\quad + \int_{\delta<|x-y|<\pi} \mid \phi_n(x-y) \mid\mid g_n(y) - g_n(x) \mid \,dy \Big) \,dx \\
&\leq \int_{\mathbb{T}} \left(\frac{\varepsilon}{\heartsuit M} \int_{|x-y|<\delta} \mid \phi_n(x-y) \mid \,dy + A \int_{\delta<|x-y|<\pi} \mid \phi_n(x-y) \mid \,dy \right) \,dx.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\int_{|x-y|<\delta} \mid \phi_n(x-y) \mid \,dy &\leq \int_{\mathbb{T}} \mid \phi_n(x-y) \mid \,dy \\
&= \int_{\mathbb{T}} \mid \phi_n(y) \mid \,dy \\
&\leq M
\end{aligned}
$$

Note:

• The equality above holds since the only thing we just did was a change a variable!
• The last inequality holds because we know that ϕ_n is an approximate identity.

Since $\lim_{n\to 0} \int_{\mathbb{T}} |\phi(x)|\,dx = 0$, then there exists an integer $N \in \mathbb{N}$ such that for no matter what $n > N$, we have that

$$
\int_{\mathbb{T}} |\phi_n(x)|\,dx < \frac{\varepsilon}{12\pi A}.
$$

Doing almost the same thing for the second integral we obtained after splitting the inner one, we get

$$
\begin{aligned}
\int_{\delta<|x-y|<\pi} |\phi(x)|\,dy &< \int_{\mathbb{T}} |\phi(x)|\,dy \\
&= \int_{\mathbb{T}} |\phi(y)|\,dy \\
&\leq \frac{\varepsilon}{12\pi A}.
\end{aligned}
$$

Putting everything we got so far together, we get that

$$\begin{aligned}
&\|\phi_n * g_n - g_n\|_{L^1(\mathbb{T})} \\
&\leq \int_{\mathbb{T}} \left(\frac{\varepsilon}{12\pi M} M + \frac{\varepsilon}{12\pi A} A \right) \, dx \\
&= \frac{\varepsilon}{6\pi} \int_{\mathbb{T}} dx \\
&= \frac{\varepsilon}{6\pi} 2\pi \\
&= \frac{\varepsilon}{3}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\phi_n * f - f\|_{L^1(\mathbb{T})} &\leq \|\phi_n * f - \phi_n * g_n\|_{L^1(\mathbb{T})} + \|\phi_n * g_n - g_n\|_{L^1(\mathbb{T})} + \|f - g_n\|_{L^1(\mathbb{T})} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}$$

Conclusion: We indeed got that $\|\phi_n * f - f\|_{L^1(\mathbb{T})} \to 0$ when $n \to \infty$, which proves our problem.

$\square$

4. (a) Let $\{a_n : n \geq 0\}$ be a sequence of non-negative real numbers such that $a_n \to 0$ as $n \to \infty$ and

$$a_{n+1} - 2a_n + a_{n-1} \geq 0.$$

Show that the series

$$\sum_{n \geq 1} n(a_{n+1} - 2a_n + a_{n-1})$$

converges to a_0. Hint: $\sum(a_{n+1} - a_n)$ is a convergent, decreasing telescoping series.

Proof. Take the sequence $\{a_n\}_n$ as said in the hypothesis: a sequence of nonzero real terms with the property

$$a_{n+1} - 2a_n + a_{n-1} \geq 0.$$

Notice that this implies that $a_n - a_{n+1} \leq a_{n-1} - a_n$ and therefore, if we define the sequence given by the general term $b_n := a_n - a_{n+1}$ then $\{b_n\}_n$ is a decreasing sequence. It is not hard to see that $\sum_{n=0} b_n = 0$. The only thing you have to do is to first compute the S_N, which is the N-th terms partial sum and then make N go to ∞. Expanding the sum $S_N$we get

$$\begin{aligned}
S_N &= \sum_{n=0}^{\infty} b_n \\
&= b_0 + b_1 + \cdots + b_N \\
&= a_0 - a_1 + \\
&\quad a_1 - a_2 + \\
&\quad a_2 - a_3 + \\
&\quad \cdots\cdots \\
&\quad a_{N-2} - a_{N-1} + \\
&\quad a_{N-1} - a_N. \\
&= a_0 - a_N.
\end{aligned}$$

Making N going to ∞, and using the hypothesis of the problem (that $a_n \to 0$ as $n \to \infty$), we get that indeed $S_N \to a_0$ as $n \to \infty$; i.e $\sum_{n=0}^{\infty} b_n = a_0$.

Now, let's try to prove that $\sum_{n=1}^{\infty} n\,(a_{n+1} - 2a_n + a_{n-1})$ converges to a_0. Spitting the sum above and computing the sum of the first N terms we get:

$$\begin{aligned}
s_N &= \sum_{n=1}^{N} n(a_{n+1} - 2a_n + a_{n-1}) \\
&= a_0 - a_N - N(a_N - a_{N+1}) \\
&= a_0 - a_N - Nb_N.
\end{aligned}$$

More explicitly:

$$\begin{aligned}
S_N &= \sum_{n=1}^{N} n\,(a_{n+1} - 2a_n + a_{n-1}) \\
&= \sum_{n=1}^{N} n\,(a_{n+1} - a_n) - \sum_{n=1}^{N} n\,(a_n - a_{n-1}) \\
&= a_0 - a_N - Nb_N.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} n(a_{n+1} - 2a_n + a_{n-1}) &= \lim_{N\to\infty} a_0 - a_N - Nb_N \\
&= a_0 - \lim_{n\to\infty} nb_n.
\end{aligned}$$

One thing that we need to show is that $\lim_{n\to\infty} nb_n = 0$.

I claim that $\lim_{n\to\infty} nb_n = 0$.

Proof of the claim:

The sequence b_n is nonnegative and decreasing, so

$$0 \leq \frac{n}{2} b_n \leq \sum_{[n/2]+1}^{n} b_k.$$

Since $\sum b_k$ converges, it is Cauchy

$$\lim_{m,n\to\infty} \sum_{k=m}^{n} b_k \to 0$$

and the result follows.

Coming back to the limit we are asked to compute, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n(a_{n+1} - 2a_n + a_{n-1}) &= \lim_{N\to\infty} a_0 - a_N - N b_N \\ &= a_0 - \lim_{n\to\infty} n b_n \\ &= a_0.\end{aligned}$$

□

(b) For $N \geq 0$, let $K_N \geq 0$ denote the Fejér kernel

$$K_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{inx}.$$

Show the series

$$f(x) = \sum_{n\geq 1} n(a_{n+1} - 2a_n + a_{n-1}) K_{n-1}(x)$$

converges in $L^1(\mathbb{T})$ to a non-negative function $f \in L^1(\mathbb{T})$ whose Fourier coefficients are $a_{|n|}$, i.e.,

$$f(x) \sim \sum_{n\in\mathbb{Z}} a_{|n|} e^{inx}.$$

Proof.

Denote by $f_N(x) := \sum_{n=1}^{N} n(a_{n+1} - 2a_n + a_{n-1}) K_{n-1}(x)$.

• First observe that $f_N \geq 0$ for all N.

• One more important comment to make is that $f_{N+1} \geq f_N$ for all N since we just add a positive term to f_N to get f_{N+1}. Therefore we are in the hypothesis of the Monotone Convergence Theorem:

$$f_N \to f \text{ pointwise and } \int f_N dx \to \int f dx$$

- Since f is the limit of positive functions, it is non-negative; also,

$$|f_N|_{L^1(\mathbb{T})} = \int_{\mathbb{T}} \mid f_N \mid \, dx = \int f_N \, dx.$$

- Similarly we get, $|f|_{L^1(\mathbb{T})} = \int_{\mathbb{T}} f \, dx.$

We conclude that, if $\lim_{N\to\infty} |f_N|_{L^1(\mathbb{T})}$ is finite, then this will implicitly give us $f \in L^1(\mathbb{T})$.

For $\forall N$ we obtain:

$$|f_N|_{L^1(\mathbb{T})} = \sum_{n=1}^{N} n(a_{n+1} - 2a_n + a_{n-1}) \sum_{j=-(n-1)}^{n-1} \left[\left(1 - \frac{|j|}{n}\right) \int_{\mathbb{T}} e^{ijx} dx \right].$$

Since

$$\int_{\mathbb{T}} e^{ijx} dx = 2\pi\delta_{0,j}$$

and

$$1 - \frac{|j|}{n} = 1,$$

when $j = 0$ we get

$$|f_N|_{L^1(\mathbb{T})} = 2\pi \sum_{n=1}^{N} n(a_{n+1} - 2a_n + a_{n-1}).$$

- Notice that

$$\sum_{n=1}^{N} n(a_{n+1} - 2a_n + a_{n-1}) \to a_0$$

as $N \to \infty$, hence

$$|f_N|_{L^1(\mathbb{T})} = 2\pi a_0$$

and also $f \in L^1(\mathbb{T})$.

Coming back to the last part of the question: we will start computing the Fourier coefficients of $f(x)$:

$$\widehat{f(j)} = \frac{1}{2\pi}\sum_{n=1}^{\infty} n(a_{n+1}-2a_n+a_{n-1}) \sum_{k=-(n-1)}^{n-1}\left(1-\frac{|j|}{n}\right)\int_{\mathbb{T}} e^{ikx}e^{-ijx}dx$$

• Now, since $\{e^{inx}\,|\,n\in\mathbb{Z}\}$ is an orthogonal set (this is easy to check), then

$$\int_{\mathbb{T}} e^{ikx}e^{-ijx}dx \neq 0$$

iff

$$-(n-1)\leq j\leq n-1$$

iff $|j|+1\leq n$.

If $n<|j|+1$ then

$$n(a_{n+1}-2a_n+a_{n-1}) \sum_{k=-(n-1)}^{n-1}\left(1-\frac{|j|}{n}\right)\int_{\mathbb{T}} e^{ikx}e^{-ijx}dx = 0.$$

• Therefore we come to the conclusion:

$$\begin{aligned}\widehat{f(j)} &= \frac{2\pi}{2\pi}\sum_{n=|j|+1}^{\infty} n(a_{n+1}-2a_n+a_{n-1}) \sum_{k=-(n-1)}^{n-1}\left(1-\frac{|j|}{n}\right)\delta_{kj} \\ &= \sum_{n=|j|+1}^{\infty} n(a_{n+1}-2a_n+a_{n-1})\left(1-\frac{|j|}{n}\right)\end{aligned}$$

The trick here is to rearrange the terms, or in other words, to shift the counting with $|j|$ step behind, so that we will recover exactly the same kind of series we just proved above that converges to a_0, but is our case since the index changed, we get that the series converges to $a_{|j|}$.

• Therefore,

$$\widehat{f(j)} = a_{|j|}$$

Hence the Fourier series of f (after renaming the j; it became n) is

$$f(x)\sim\sum_{n\in Z} a_{|n|}e^{inx}.$$

□

(c) Show there is a function $f \in L^1(\mathbb{T})$ such that

$$f(x) \sim \sum_{|n|\geq 2} \frac{1}{\log |n|} e^{inx}.$$

Proof.
• This question is clearly based on the the facts we have just proved in (b).In order to prove that such an L^1 function exists, it is sufficient to show that the sequence $a_n = \dfrac{1}{\ln(n)}$ for $n \geq 2$ satisfies the properties listed in part (a). This will be trivial to check.

Checking that

$$a_{n+1} - 2a_n + a_{n-1} \geq 0$$

holds, for the a_n we just defined translates to $a_n - a_{n+1} \leq a_{n-1} - a_n$ which is equivalent to

$$\frac{1}{\ln(n)} - \frac{1}{\ln(n+1)} \leq \frac{1}{\ln(n)} - \frac{1}{\ln(n-1)}.$$

• Suffices to show that the function we define being

$$m(x) := \frac{1}{\ln(x)} - \frac{1}{\ln(x+1)}$$

is a decreasing function on the interval $[2, \infty)$. To see this we use calculus; compute the derivative $m^{'}(x)$ set it zero, and try to find the extrema points. Then construct the chart (or using the second derivative test) and see that indeed the relation from part (a) is satisfied(let me know if you have trouble finishing up those details!).

□

(d) Suppose that $f \in L^1(\mathbb{T})$ has imaginary Fourier coefficients $\{ib_n : n \in \mathbb{Z}\}$ such that $b_n \geq 0$ for $n \geq 0$ and $b_{-n} = -b_n$. Show that

$$\sum_{n\geq 1} \frac{b_n}{n}$$

converges.
Hint: The integral

$$F(x) = \int_0^x f(t)dt$$

is a continuous function (in fact, absolutely continuous) with Fourier coefficients

$$\frac{1}{2\pi} \int F(x)e^{-inx}dx = \frac{b_n}{n} \qquad \text{for } n \neq 0.$$

Use the fact that $K_N * F(0)$ converges to $F(0)$ since $\{K_N\}$ is an approximate identity.

Proof. From Amanda's .tex file

For $n \neq 0$ the Fourier coefficients of $F(x)$ are defined to be:

$$\widehat{F(n)} = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^x f(t)dt \right) e^{-inx} dx$$

Using integration by parts with $u = \int_0^x f(t)dt$ and $dv = e^{-inx}dx$ we get that

$$\begin{aligned} \widehat{F(n)} &= \frac{e^{-inx}}{in2\pi} \int_0^x f(t)dt\Big|_0^{2\pi} + \frac{1}{in2\pi} \int_0^{2\pi} f(x)e^{-inx}\, dx \\ &= \frac{\widehat{f(n)}}{in} = \frac{ib_n}{in} = \frac{b_n}{n}. \end{aligned}$$

Since F is a continuous function, we get that F is equals its Fourier series. Therefore

$$F(0) = \widehat{F(0)} + \sum_{n \neq 0} \frac{b_n}{n}$$

Since $b_{-n} = -b_n$ we see that

$$\frac{b_{-n}}{-n} = \frac{-b_n}{-n} = \frac{b_n}{n}$$

Hence

$$\sum_{n=-1}^{-\infty} \frac{b_n}{n} = \sum_{n=1}^{\infty} \frac{b_{-n}}{-n} = \sum_{n=1}^{\infty} \frac{b_n}{n}.$$

- So

$$F(0) = \widehat{F(0)} + 2\sum_{n=1}^{\infty} \frac{b_n}{n}$$

Since $F(0) = \displaystyle\int_0^0 f(t)dt = 0$ it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{b_n}{n} = \frac{-\widehat{F(0)}}{2}$ and therefore the sum converges. ∎

Proof.

- If there were such a function $f(x)$, its Fourier series coefficients $\frac{i \operatorname{sgn} n}{\log |n|}$

satisfy the conditions in part d. Hence by the contra-positive of (d), we need to show that

$$\sum_{n=2}^{\infty} \frac{\operatorname{sgn}(n)}{n \log |n|} = \sum_{n=2}^{\infty} \frac{1}{n \log n}$$

is a divergent series.

• This is a series with positive terms and therefore using your experience gained in working with series, you can see that the right convergence test to use will be integral test (a quicker one will be the comparison test).

• We get that the series diverges. Hence we conclude that we cannot find a function $f \in L^1(\mathbb{T})$ with this Fourier expansion given in the hypothesis of part d.

$\square$