201B, Winter '11, Professor John Hunter
Homework 3 Solutions

1. Suppose that $\sum_{n=0}^{\infty} c_n$ is a series of complex numbers with partial sums

$$s_n = \sum_{k=0}^{n} c_k.$$

The series is Borel summable with Borel sum s if the following limit exists:

$$s = \lim_{x\to+\infty} e^{-x} \left(\sum_{n=0}^{\infty} \frac{s_n x^n}{n!} \right).$$

(a) If the series $\sum_{n=0}^{\infty} c_n = s$ is convergent, show that it is Borel summable with Borel sum equal to s, i.e. Borel summation is regular.

Proof. Let $\varepsilon > 0$ be given. Choose N such that $|s_n - s| < \varepsilon$ for $n \geq N$. Now,

$$\begin{aligned}
\lim_{x\to+\infty} e^{-x} \left(\sum_{n=0}^{\infty} \frac{s_n x^n}{n!} \right) &= \lim_{x\to+\infty} e^{-x} \left(\sum_{n=0}^{N} \frac{s_n x^n}{n!} + \sum_{n=N+1}^{\infty} \frac{s_n x^n}{n!} \right) \\
&= \lim_{x\to+\infty} e^{-x} \left(\sum_{n=0}^{N} \frac{s_n x^n}{n!} \right) + \lim_{x\to+\infty} e^{-x} \left(\sum_{n=N+1}^{\infty} \frac{s_n x^n}{n!} \right) \\
&= \lim_{x\to+\infty} e^{-x} \left(\sum_{n=N+1}^{\infty} \frac{s_n x^n}{n!} \right).
\end{aligned}$$

The reaon why the first term went away is because $\sum_{n=0}^{N} \frac{s_n x^n}{n!}$ is just a constatnt, and when we multiply it by e^{-x} and then take the limit as $x \to \infty$, we get zero.
So, now we have that

$$\begin{aligned}
\left| s - \lim_{x\to+\infty} e^{-x} \left(\sum_{n=N+1}^{\infty} \frac{s_n x^n}{n!} \right) \right| &= \left| \lim_{x\to+\infty} e^{-x} \left(\sum_{n=0}^{\infty} \frac{s x^n}{n!} - \sum_{n=N+1}^{\infty} \frac{s_n x^n}{n!} \right) \right| \\
&= \left| \lim_{x\to+\infty} e^{-x} \left(\sum_{n=0}^{\infty} \frac{(s - s_n) x^n}{n!} + \sum_{n=0}^{N} \frac{s_n x^n}{n!} \right) \right| \\
&\leq |s - s_n| \lim_{x\to+\infty} \left| e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} \right| + \lim_{x\to+\infty} e^{-x} \left| \sum_{n=0}^{N} \frac{s_n x^n}{n!} \right| \\
&< \varepsilon(1) + 0 = \varepsilon.
\end{aligned}$$

Then, since our choice of ε was arbitrary, we must have that Borel summation is regular.

$\square$

(b) For what complex numbers $a \in \mathbb{C}$ is the geometric series

$$\sum_{n=0}^{\infty} a^n$$

Borel summable? What is its Borel sum? For what $a \in \mathbb{C}$ is this series Cesàro summable? Abel summable?

Proof. See remarks.

Borel Summability

By part a), we know that this sum will converge if $|a| < 1$, and from part a) we know that the limit is $\dfrac{1}{1-a}$.
Note that if $a = 1$ we have

$$\begin{aligned}
\lim_{x\to+\infty} e^{-x} \sum_{n=0}^{\infty} \frac{nx^n}{n!} &= \lim_{x\to+\infty} e^{-x} \sum_{n=1}^{\infty} \frac{nx^n}{(n)!} \\
&= \lim_{x\to+\infty} e^{-x} \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} \\
&= \lim_{x\to+\infty} e^{-x} \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n)!} \\
&= \lim_{x\to+\infty} x = \infty.
\end{aligned}$$

Now, suppose $a \neq 1$ and therfore we can write a in its general form $a = u + iv$. Then $s_n = \frac{1-a^{n+1}}{1-a}$.
So,

$$\begin{aligned}
\lim_{x\to+\infty} e^{-x} \sum_{n=0}^{\infty} \frac{1-a^{n+1}}{1-a} \frac{x^n}{n!} &= \frac{1}{1-a} \lim_{x\to+\infty} e^{-x} \sum_{n=0}^{\infty} (1-a^{n+1}) \frac{x^n}{n!} \\
&= \frac{1}{1-a} \left(1 - a \lim_{x\to+\infty} e^{-x} \sum_{n=0}^{\infty} \frac{a^n x^n}{n!} \right) \\
&= \frac{1}{1-a} \left(1 - a \lim_{x\to\infty} e^{(a-1)x} \right) \\
&= \frac{1}{1-a} \left(1 - a \lim_{x\to\infty} e^{(u+iv-1)x} \right) \\
&= \frac{1}{1-a} \left(1 - a \lim_{x\to\infty} e^{(u-1)x} e^{ivx} \right),
\end{aligned}$$

from which we see that the sum will be finite so long as $\lim_{x\to\infty} e^{(u-1)x} = 0$, which happens when $u < 1$ i.e., when $\operatorname{Re}[a] < 1$.

Cesàro Summability

Using Cesàro summation, we see

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} \frac{1-a^{k+1}}{1-a}}{n+1} = \frac{1}{1-a} \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} 1 - a^{k+1}$$
$$= \frac{1}{1-a} \left(1 - \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} a^{k+1} \right).$$

Note that $\sum_{k=0}^{n} a^{k+1}$ is a geometric series, and there are some details that need to be discussed in order to conclude the convergence of it. Hence,we see that this sum will converge if $|a| \leq 1$ and $a \neq 1$. So the convergence of the whole expression above will be convergent if $|a| \leq 1$ and $a \neq 1$.

Abel Summability

Using Abel summation, we see

$$\lim_{r\to 1^-} \lim_{n\to\infty} \sum_{k=0} a^k r^k = \lim_{r\to 1^-} \sum_{n} a^n r^n$$
$$= \lim_{r\to 1^-} \sum_{n} (ar)^n$$

For $a > 1$, and for r that approches 1 from the lower side, we can observe that we cannot have $ar > 1$. Since the above series is a geometric series, we can conclude that in this case, i.e., $ar > 1$, the sum is divergent.
It will converge as long as $|ra| \leq 1$, which is going to happen when $|a| = 1$ and $r < 1$. The sum will converge to $\lim_{r\to 1^-} \frac{1}{1-ar}$. Hence, as long as $a \neq 1$, we have convergece of the series.

□

(c) Do you get anything useful from the Borel summation of a Fourier series?

Proof. See remarks.

□

2. Let $A(\mathbb{T})$ denote the space of integrable functions whose Fourier coefficients are absolutely convergent. That is, $f \in A(\mathbb{T})$ if

$$\sum_{n \in \mathbb{Z}} \left| \hat{f}(n) \right| < \infty.$$

(a) If $f \in A(\mathbb{T})$, show that $f \in C(\mathbb{T})$. Also show that $f \in A(\mathbb{T})$ if and only if $f = g * h$ for some functions $g, h \in L^2(\mathbb{T})$.

Proof. Recall that we showed in Homework 1 that $f * g \in C(\mathbb{T})$ for $f, g \in L^2(\mathbb{T})$. Then we need only show that if $f \in A(\mathbb{T})$, then $f = g * h$ for $g, h \in L^2(\mathbb{T})$. So, let $f \in A(\mathbb{T})$. Then

$$\sum_{n \in \mathbb{Z}} \left| \hat{f}(n) \right| < \infty.$$

Define the functions

$$g(z) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sqrt{|\hat{f}(n)|} e^{-inz} \quad \text{and}$$
$$h(z) = \sum_{n \in \mathbb{Z}} \sqrt{|\hat{f}(n)|} e^{-inz} e^{i \, arg \, z}.$$

Then these functions are in $L^2(\mathbb{T})$ since the coefficients of f are absolutely convergent.
Now,

$$\begin{aligned}
g * h &= \int_{\mathbb{T}} g(z - w) h(w) \, dw \\
&= \int_{\mathbb{T}} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sqrt{|\hat{f}(n)|} e^{-in(z-w)} \sum_{m \in \mathbb{Z}} \sqrt{|\hat{f}(m)|} e^{-imw} e^{i \, arg \, z} \, dw \\
&= \frac{1}{2\pi} \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} |\hat{f}(n)| e^{-in(z-w)} e^{-inw} e^{i \, arg \, z} \, dw \\
&= \frac{1}{2\pi} \int_{\mathbb{T}} \sum_{n \in \mathbb{Z}} |\hat{f}(n)| e^{i \, arg \, z} e^{-inz} \, dw \\
&= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{-inz} \int_{\mathbb{T}} dw \\
&= \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{-inz} \\
&= f(z)
\end{aligned}$$

Now, suppose that $f = g * h$ for $g, h \in L^2(\mathbb{T})$. Then $\hat{f}(n) = \hat{g}(n)\hat{h}(n)$.

So,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}|\hat{f}(n)| &= \sum_{n\in\mathbb{Z}}|\hat{g}(n)\hat{h}(n)| \\
&= \sum_{n\in\mathbb{Z}}|\hat{g}(n)|\,\overline{|\hat{h}(n)|} \\
&= \left\langle |\hat{g}(n)|, |\hat{h}(n)| \right\rangle \le \||\hat{g}(n)|\|_2 \||\hat{h}(n)|\|_2 < \infty.
\end{aligned}$$

□

(b) If $f, g \in A(\mathbb{T})$, show that $fg \in A(\mathbb{T})$ and express $\hat{f}g$ in terms of $\hat{f}, \hat{g}$.

Proof.

$$\begin{aligned}
\widehat{fg}(n) &= \frac{1}{2\pi}\int_{\mathbb{T}} f(x)g(x)e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \sum_{k\in\mathbb{Z}}\hat{f}(k)e^{ikx}\sum_{m\in\mathbb{Z}}\hat{g}(m)e^{imx}e^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \sum_{k,m}\hat{f}(k)\hat{g}(m)e^{i(k+m)x}e^{-inx}\,dx \\
&= \frac{1}{2\pi}\sum_{k,m}\hat{f}(k)\hat{g}(m)\int_{\mathbb{T}} e^{i(k+m-n)x}\,dx \\
&= \frac{1}{2\pi}\sum_{k+m-n=0}\hat{f}(k)\hat{g}(m).
\end{aligned}$$

Note: We used the Lebesgue's Dominated Convergence Theorem, in order to switch the integral ans the sum.
Now, what remais to prove is that they are absolutely summable. For this let's takethe sum over all n, and get:

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{k+m-n=0}|\hat{f}(k)\hat{g}(m)| = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\hat{f}(k)\hat{g}(n-k)|\sum_{n\in\mathbb{Z}}|\hat{f}(n)|\sum_{k\in\mathbb{Z}}|\hat{g}(k)| < \infty.$$

□

(c) Give an example of a function $f \in C(\mathbb{T})$ such that $f \notin A(\mathbb{T})$.

Proof.

□

3. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc in the complex plane. The Hardy space $H^2(\mathbb{D})$ is the space of functions with a power series expansion

$$F(z) = \sum_{n=0}^{\infty} c_n z^n \tag{0.1}$$

such that

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty. \tag{0.2}$$

This is a Hilbert space with inner product

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle = \sum_{n=0}^{\infty} \overline{a}_n b_n.$$

(a) If (0.1) holds, show that the power series (0.2) converges in $\mathbb{D}$ to a holomorphic function $F : \mathbb{D} \to \mathbb{C}$.

Proof. Since

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty,$$

then there exists an $N \in \mathbb{N}^*$ such that for all the $n > N$, $c_n \leq 1$.
Let $\varepsilon > 0$ be given. Consider

$$F(1-\varepsilon) = \sum_{n=0}^{\infty} c_n (1-\varepsilon)^n.$$

Then note that

$$\begin{aligned} |F(1-\varepsilon)| &\leq \sum_{n=0}^{\infty} |c_n|(1-\varepsilon)^n \\ &= \sum_{n=0}^{N} |c_n|(1-\varepsilon)^n + \sum_{n=N+1}^{\infty} |c_n|(1-\varepsilon)^n \\ &\leq \sum_{n=0}^{N} |c_n|(1-\varepsilon)^n + \sum_{n=N+1}^{\infty} (1-\varepsilon)^n \\ &\leq \infty. \end{aligned}$$

Since F at $(1-\varepsilon)$ can be represented by a powers series centered at 0, then we get that F is analytic on the ball centered at 0 and of radius $(1-\varepsilon)$ i.e., $B_{1-\varepsilon}(0)$. Since the $\varepsilon > 0$ was arbitrarly choosen, we conclude that F is analytic on D.

□

(b) Is $\dfrac{1}{1-z} \in H^2(\mathbb{D})$? If $\theta_0 \in \mathbb{T}$, give an example of a function $F \in H^2(\mathbb{D})$ which does not extend to a function that is analytic at $z = e^{i\theta_0}$.

Proof. Note that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

and

$$\sum_{n=0}^{\infty} 1 = \infty,$$

so

$$\frac{1}{1-z} \notin H^2(\mathbb{D}).$$

Note that taking the integral of

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

gives us

$$-\log(1-z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}.$$

So this gives us a function in $H^2(\mathbb{D})$ which is not analytic at $z = 1$. From this, we see that for any $\theta_0 \in \mathbb{T}$ the function $-\log(\theta_0 - z)$ doesn't extend to an analytic function.

□

(c) If $F \in H^2(\mathbb{D})$, show that

$$\|F\|_{H^2}^2 := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 \, d\theta < \infty.$$

Show conversely that if $F : \mathbb{D} \to \mathbb{C}$ is a holomorphic function such that $\|F\|_{H^2}^2 < \infty$ then $F \in H^2(\mathbb{D})$.

Proof. Due to Tim.

$$
\begin{aligned}
\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^2 \, d\theta &= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} r^n e^{in\theta} c_n \right|^2 d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} r^n e^{in\theta} c_n \right| \left| \sum_{n=0}^{\infty} r^n e^{-in\theta} \overline{c}_n \right| d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi} \int_{\mathbb{T}} \left(\sum_{m,n \in \mathbb{Z}} r^{n+m} c_m \overline{c}_n e^{i(m-n)\theta} \right) d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi} \sum_{m,n} 2\pi \delta_{m,n} c_m \overline{c}_n r^{n+m} \\
&= \sup_{0<r<1} \sum_{n \in \mathbb{Z}} |c_n|^2 r^{2n} \\
&= \sum_{n \in \mathbb{Z}} |c_n|^2 \\
&= \langle F, F \rangle.
\end{aligned}
$$

Note that since holomorphic functions have power series expansions, the above calculation also establishes the converse. □

(d) Let

$$\tilde{H}^2(\mathbb{T}) := \left\{ f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0 \right\}.$$

If $F \in H^2(\mathbb{D})$ is given by (1) and $0 < r < 1$, define $f_r \in L^2(\mathbb{T})$ by

$$f_r(\theta) = F(re^{i\theta}).$$

Show that $f_r \to f$ as $r \to 1^-$ in $L^2(\mathbb{T})$ where

$$f(\theta) = \sum_{n=0}^{\infty} c_n e^{in\theta} \in \tilde{H}^2(\mathbb{T}).$$

Conversely, if $f \in \tilde{H}^2(\mathbb{T})$, define $F : \mathbb{D} \to \mathbb{C}$ by

$$F(re^{i\theta}) = (P_r * f)(\theta)$$

where P_r is the Poisson kernel. Show that $F \in H^2(\mathbb{T})$.

Proof. Due to Tim.

$$\begin{aligned}
P_r * f(x) &= \int P_r(x-y) f(y)\, dy \\
&= \frac{1}{2\pi} \int \sum_{n \in \mathbb{Z}} r^{|n|} e^{in(x-y)} \sum_{n=0}^{\infty} c_n e^{iny}\, dy \\
&= \frac{1}{2\pi} \int \sum_{\substack{n \in \mathbb{Z} \\ m \geq 0}} r^{|n|} e^{in(x-y)} c_m e^{imy}\, dy \\
&= \frac{1}{2\pi} \int \sum_{\substack{n \in \mathbb{Z} \\ m \geq 0}} r^{|n|} e^{in(x-y)+imy} c_m\, dy \\
&= \sum_{n \geq 0} r^n e^{iny} c_n\, dy \\
&= f_r(x).
\end{aligned}$$

Then, since P_r is an approximate identity, we must have that $f_r \to f$. $\square$