**201B, Winter '11, Professor John Hunter**
**Homework 4 Solutions**

1. Let $D \in \mathbb{R}^2$ be the unit disc and $f \in C(\partial D)$ a continuous function defined on the unit circle $\partial D$. Suppose that $u : \overline{D} \to \mathbb{R}$ is a function $u \in C^2 \cap C(\overline{D})$ such that

$$\begin{cases} \Delta u = 0 \text{ in } D \\ u = f \text{ on } \partial D \end{cases}$$

(a) Show that

$$\max_{\overline{D}} u = \max_{\partial D} f.$$

*Proof.* **a)** This part is in fact a theorem often used in PDE's called the "Weak Maximal Principle". There are at least two ways of proving this result, but I will write up the one that uses the hint already given to us. The other way of proving it, is by using the Mean Value Property Theorem.
Consider the function

$$u^{\varepsilon}(x, y) = u(x, y) + \varepsilon(x^2 + y^2).$$

Applying the differential operator, $\triangle$, we get that

$$\triangle u^{\varepsilon}(x, y) = \triangle u(x, y) + \varepsilon \triangle (x^2 + y^2) = 4\varepsilon > 0.$$

Here, I used the fact that $u$ is harmonic on the unit disk $\mathbb{D}$, and therefore the $\triangle u = 0$. Hence, if $u^{\varepsilon}$ attains a maximum at an interior point of $\mathbb{D}$ then $\triangle u^{\varepsilon}$ needs to be less or equal to zero–it is not much more than the second order derivative test that we need to be satisfied in order to have a maximum. But, $\triangle u^{\varepsilon} = 4\varepsilon \geq 0$, and thus $u^{\varepsilon}$ has no interior maximum and it attains its maximum on the boundary.
If $x^2 + y^2 < 1$, then

$$\sup_{\mathbb{D}} u \leq \sup_{\mathbb{D}} u^{\varepsilon} \leq \sup_{\partial \mathbb{D}} u^{\varepsilon} \leq \sup_{\partial \mathbb{D}} u + \varepsilon.$$

Letting $\varepsilon \to 0$, we get the desired result.

□

(b) Deduce that a solution of given problem is unique and is therefore given by

$$u(r, \theta) = (P_r * f)(\theta)$$

in $0 \leq r < 1$ where $P_r$ is the Poisson kernel.

*Proof.* **b)**

- **Uniqueness**

Assume that we have two solutions, $u_1$ and $u_2$, satisfying the Dirichlet boundary problem then

$$\begin{cases} \triangle u_1 = 0 & \text{in } \mathbb{D} \\ \triangle u_2 = 0 & \text{in } \mathbb{D} \\ u_1 = f & \text{on } \partial\mathbb{D} \\ u_2 = f & \text{on } \partial\mathbb{D} \end{cases}$$

Substracting the we get first that

$$\begin{cases} \triangle(u_1 - u_2) = 0 & \text{in } \mathbb{D} \\ \triangle(u_2 - u_1) = 0 & \text{in } \mathbb{D} \\ u_1 - u_2 = 0 & \text{on } \partial\mathbb{D} \\ u_2 - u_1 = 0 & \text{on } \partial\mathbb{D} \end{cases}$$

which, by part **a** of the problem, tell you that since $u_1 - u_2$ is harmonic in $\mathbb{D}$, then the maximum value of $u_1 - u_2$ in $\mathbb{D}$ must occur at the boundary of $\mathbb{D}$–by the maximal principle. But 0 is the only value of $u_1 - u_2$ on the boundary, so the maximum of $u_1 - u_2$ in $\mathbb{D}$ must be 0, i.e., $u_1 - u_2 \leq 0$ . The same kind of logic we use for the $u_2 - u_1$ and we get that $u_2 - u_1 \leq 0$. Putting together what we got, we have that in order for the last two inequalities to hold in the same time we need $u_1 - u_2 = 0$, which translates to $u_1 = u_2$ in $\mathbb{D}$. Therefore, solutions to the Dirichlet problem with continuous boundary data are unique.

- **Separation of variable to find solutions**

We can solve the Dirichlet problem on a disk in the plane, using separation of variables. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$ be the unit disk. It will actually be more convenient to work in polar coordinates $(r, \theta)$. In polar coordinates, the unit disk is given by

$$\mathbb{D} = \{(r, \theta) \,|\, 0 \leq r \leq 1 \text{ and } 0 \leq \theta < 2\pi\}\,.$$

Notice that every point in the disk has a unique representation $(r, \theta)$ as defined above, except for the origin which can be written as $(0, \theta)$ for any choice of angle $0 \leq \theta < 2\pi$.

Suppose that $u(r, \theta) = R(r)\Theta(\theta)$ is a harmonic function in $\mathbb{D}$-this method is called separation of variable for PDE's. We want to find ODEs that $R$ and $\Theta$ satisfy. Since $u$ is harmonic, $\triangle u = u_{xx} + u_{yy} = 0$. But this form of Laplaces equation is not helpful, since it consists derivatives with respect to $x$ and $y$, while we want derivatives with respect to $r$ and $\theta$. Hence we must change of variables.

Recall that $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r = \sqrt{x^2+y^2}$ and $\theta = \tan^1(\frac{x}{y})$. Then

$$\begin{aligned}
\partial_x r &= \frac{x}{r} = \cos(\theta),\\
\partial_y r &= \frac{x}{r} = \sin(\theta),\\
\partial_x \theta &= \frac{y}{r^2} = \frac{1}{r}\sin(\theta)
\end{aligned}$$

and $\partial_y \theta = \frac{x}{r^2} = \frac{1}{r}\cos(\theta)$.

By the chain rule, it follows that

$$\begin{aligned}
u_x &= u_r \cos(\theta) u_\theta \frac{\sin(\theta)}{r},\\
u_{xx} &= \left(u_{rr}\cos(\theta) u_{r\theta}\frac{\sin(\theta)}{r}\right)\cos(\theta) + u_r \frac{\sin^2(\theta)}{r}\\
&\quad - \left(u_{\theta r}\cos(\theta) u_{\theta\theta}\frac{\sin(\theta)}{r}\right)\frac{\sin(\theta)}{r} + 2u_\theta \frac{\cos(\theta)\sin(\theta)}{r^2}
\end{aligned}$$

and

$$\begin{aligned}
u_y &= u_r \sin(\theta) + u_\theta \frac{\cos(\theta)}{r},\\
u_{yy} &= \left(u_{rr}\sin(\theta) + u_{r\theta}\frac{\cos(\theta)}{r}\right)\sin(\theta) + u_r \frac{\cos^2(\theta)}{r}\\
&\quad + \left(u_{\theta r}\sin(\theta) u_{\theta\theta}\frac{\cos(\theta)}{r}\right)\frac{\cos(\theta)}{r} - 2u_\theta \frac{\cos(\theta)\sin(\theta)}{r^2}.
\end{aligned}$$

Adding the formulas we found for $u_{xx}$ and $u_{yy}$ and simplifying the expression we get:

$$\begin{aligned}
u_{xx} + u_{yy} =& u_{rr}(\sin^2(\theta) + \cos^2(\theta)) + u_r(\frac{\sin^2(\theta)}{r}\\
&+ \frac{\cos^2(\theta)}{r}) + u_{\theta\theta}(\frac{\sin^2(\theta)}{r^2} + \frac{\cos^2(\theta)}{r^2}).
\end{aligned}$$

Thus the Laplacian in polar coordinates is given by:

$$\triangle u = u_{rr} + r^{1-} u_r + r^{-2} u_{\theta\theta}.$$

If $u(r,\theta) = R(r)\Theta(\theta)$ is harmonic, then Laplace's equation in polar coordinates, after plugging in the expression of $u$ gives us:

$$R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta'' = 0.$$

Rewriting the above equation, we get the folllwing system of ODE's:

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta}.$$

Therefore since the LHS only depends on the variable $r$ and the RHS only depends on the variable $\theta$, both sides must be equal to a constant, for example $\lambda$:

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

Hence, we end up with the following ODE's that we need to solve:

$$r^2R'' + rR' - \lambda R = 0 \quad \text{and} \quad \Theta'' + \lambda\Theta = 0.$$

I will start solving the first easy ODE, namely $\Theta'' + \lambda\Theta = 0$.. One of the things that we need to look for is the periodic solutions of period $2\pi$of this ODE. Thus we only need to find the solutions $\Theta(\theta)$ of the differential equation $\Theta'' + \lambda\Theta = 0$ which have period $2\pi$.
**There are three cases.**

**CASE I.** Suppose that $\lambda = \mu^2 << 0$. Then we know $\Theta(\theta) = c_1e^{\mu\theta} + c_2e^{-\mu\theta}$. This function can be periodic only if $c_1 = c_2 = 0$. In this case $\Theta(\theta)0$ and hence $u \equiv 0$. This case is not interesting.

**CASE II.** Suppose that $\lambda = 0$. Then $\Theta(\theta) = c_1 + c_2\theta$, which is periodic only if $c_2 = 0$. Hence $\Theta(\theta) = c_1$ is constant. When $\lambda = 0$, the differential equation for $R$ becomes $r^2R'' + rR' = 0$. Thus $R(r) = k_1 + k_2 \log r$ is the general solution of $r^2R'' + rR' = 0$. But since we want $u$ to be real-valued (finite) at the origin, we cannot allow $R(r)$ to have a $\log r$ term, i.e. we need $k_2 = 0$. Therefore, $u(r, \theta) = R(r)\Theta(\theta) = k_1c_1$ is a constant function when $\lambda = 0$.

**CASE III.** Suppose that $\lambda = \mu^2 > 0$. Then $\Theta(\theta) = c_1 \cos(\mu\theta) + c_2 \sin(\mu\theta)$ and $R(r) = k_1r^{\mu} + k_2r^{\mu}$. In order for $\Theta$ to be periodic of period $2\pi$, we need $\mu = n$ to be a positive integer. In order for $u$ to be real-valued at the origin, we cannot allow $R(r)$ to have a $r^{\mu}$ term, i.e. we must take $k_2 = 0$. Therefore, $u(r, \theta) = R(r)\Theta(\theta) = k_1r^n(c_1 \cos(n\theta) + c_2 \sin(n\theta)$ for some positive integer $n$, when $\lambda > 0$.

Therefore all harmonic functions in the disk of the form $u(r, \theta) = R(r)\Theta(\theta)$ are either constant or $u(r, ) = r^n(c_1 \cos(n\theta) + c_2 \sin(n\theta))$for some positive integer $n$. These are *the fundamental solutions* of Laplaces equation in the unit disk.

• Now lets return to solving the Dirichlet problem on the unit disk. Let $f(\theta)$ be a continuous function defined on $\partial\mathbb{D} = \{(1, \theta) : 0 \le \theta < 2\pi\}$ . We want to solve the BVP

$$\begin{cases} \triangle u = 0 & \text{in } \mathbb{D} \\ u = f & \text{on } \partial\mathbb{D} \end{cases}$$

Following the same approch as probably you saw in the 1D heat equation case or in 1D wave equation case, we can try to find $u$ which is a (infinite) linear combination of fundamental solutions of Laplaces equation. Let

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

Then $u$ is harmonic in $\mathbb{D}$. Then $u(r,\theta)$ is a harmonic extension of $f(\theta)$ to $\mathbb{D}$ provided that

$$u(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)) = f(\theta). \tag{0.1}$$

Thus we can solve the Dirichlet problem in the unit disk with boundary data $f$ provided that $f(\theta)$ has a Fourier series expansion with $L = \pi$. Summarizing what we did so far, we conclude that the solution of the Dirichlet boundary problem is given by the (0.1), where,

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t)\cos(nt)\, dt \quad \text{and} \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(t)\sin(nt)\, dt.$$

It is now possible to say that we have solved the Dirichlet problem on the unit disk, but if we work a little bit more, we can find diffrent solution from the ones we just found. This extra work will involve complex numbers, but nothing more than Eulers formula $e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$. For each integer $n$ (positive, zero or negative), define a new coefficient $c_n$ by

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t)\left(\cos(nt) - i\sin(nt)\right)\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt$$

Then we can relate the coefficients $c_n$ to the coefficients $a_n$ and $b_n$ by

$$a_0 = 2c_0\ , \quad a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}) \quad \text{for all } n \geq 1.$$

Now we can rewrite the Fourier series for $f$ in terms of $c_n$,

$$
\begin{aligned}
f(\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)) \\
&= c_0 + \sum_{n=1}^{\infty} ((c_n + c_{-n}) \cos(n\theta) + i(c_n - c_{-n}) \sin(n\theta)) \\
&= c_0 + \sum_{n=1}^{\infty} \left(c_n e^{in\theta} + c_{-n} e^{-in\theta}\right) \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\theta} + \sum_{n=-\infty}^{-1} c_n e^{in\theta} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{in\theta}.
\end{aligned}
$$

In the last line the infinite sum ranges over all integers $n$ (positive, zero and negative).
Similarly

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$

where

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} \, dt.$$

We can check that indeed the boundary condition is satisfied. Next we define and examine the function

$$P(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

First observe that we can rewrite $P(r,\theta)$ as two geometric series:

$$P(r,\theta) = \sum_{n=0}^{\infty} \left(re^{i\theta}\right)^n + \sum_{n=1}^{\infty} \left(re^{-i\theta}\right)^n.$$

Since $|re^{i\theta}| = r < 1$ and $|re^{-i\theta}| = r < 1$ these series converge and

$$
\begin{aligned}
P(r,\theta) &= \frac{1}{1 - re^{i\theta}} + \frac{1}{1 - re^{-i\theta}} - 1 \\
&= \frac{1 - r^2}{1 - r(e^{i\theta} + e^{-i\theta}) + r^2} \\
&= \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2}.
\end{aligned}
$$

Hence, by the convolution theorem $(7.28) - HN$ we proved what we were asked to prove. $\square$

2. Define $f \in L^2(\mathbb{T})$ by

$$f(x) = |x| \qquad \text{for } |x| < \pi.$$

Show that $f \in H^1(\mathbb{T})$ and compute its weak derivative $f' \in L^2(\mathbb{T})$. Is $f' \in H^1(\mathbb{T})$? For what values of $s > 0$ is it true that $f \in H^s(\mathbb{T})$?

*Proof.* • To show that $f \in H^1(\mathbb{T})$ we need to use the definition what it means for a function to belong to $H^1(\mathbb{T})$. For this we use the Fourier coefficients of $f$ that we computed in Homework 1,

$$\hat{f}(n) = \frac{2}{\pi n^2}((-1)^n - 1).$$

Hence, computing the norm of $f$, we get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} n^2|\hat{f}|^2 &= \sum_{n=\text{odd},\, n\in\mathbb{Z}} n^2 \frac{8}{\pi^2 n^4} \\ &= \sum_{n=\text{odd},\, n\in\mathbb{Z}} \frac{8}{\pi^2 n^2} < \infty\end{aligned}$$

since that is just a constant multiplied with a $p$-series with $p = 2$.
Hence, we conclude $f \in H^1(\mathbb{T})$.

• We compute the weak derivative of $f$. The weak derivative $f'$ is this unique element of $L^2(\mathbb{T})$ that by the definition satisfies

$$\int_{\mathbb{T}} f'\varphi dx = -\int_{\mathbb{T}} f\varphi' dx$$

for all $\varphi \in C^1(\mathbb{T})$. Let's compute the weak derivative of $f$ :

$$\begin{aligned}-\int_{-\pi}^{\pi} |x|\varphi'(x)\,dx &= -\int_0^{\pi} x\varphi'(x)\,dx + \int_{-\pi}^{0} x\varphi'(x)\,dx \\ &= -x\varphi(x)\Big|_0^{\pi} + \int_0^{\pi}\varphi(x)\,dx + x\varphi(x)\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\varphi(x)\,dx \\ &= -\pi\varphi(\pi) + \pi\varphi(-\pi) + \int_{-\pi}^{\pi}\operatorname{sgn}(x)\varphi(x)\,dx \\ &= \int_{-\pi}^{\pi}\operatorname{sgn}(x)\varphi(x)\,dx.\end{aligned}$$

We conclude that $\operatorname{sgn}(x)$ is the weak derivative of $f(x) = |x|$.

• Now we want to see if $f'$ is an element of the space $H^1(\mathbb{T})$. Look at the following sum:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} n^2|\hat{f}'(n)|^2 &= \sum_{n\in\mathbb{Z}} n^2 n^2 |\hat{f}(n)|^2 \\ &= \sum_{n=\text{odd},\, n\in\mathbb{Z}} \frac{8n^2}{\pi^2 n^2} \\ &= \infty.\end{aligned}$$

We can conclude that $f'$ is not an element of $H^1(\mathbb{T})$.

• To answer to the last question we see from our previous work that in order to have $\sum_{\mathbb{Z}} n^{2s}|\hat{f}(n)|^2 < \infty$, i.e., convergent, we need $\sum_{n\in\mathbb{Z}} \frac{1}{n^{4-2s}} < \infty$ which, by the $p$-series test, implies $4-2s > 1$ or equivalently $s < \frac{3}{2}$.

□

3. Suppose that $f : [0, L] \to \mathbb{R}$ is a smooth function, i.e. $f \in C^1([0, L])$ such that $f(0) = f(L) = 0$. Prove that

$$\int_0^L |f(x)|^2 dx \leq \left(\frac{L}{\pi}\right)^2 \int_0^L [f'(x)]^2\, dx.$$

Show that the constant in this inequality is sharp. Why do you need to assume that $f(0) = f(L) = 0$? Show that you cannot estimate the $L^2$-norm of a smooth, square-integrable function $f : [0, \infty) \to \mathbb{R}$ such that $f(0) = 0$ in terms of the $L^2$ norm of its derivative.

*Proof.* I will write a slightly different proof than the one I gave in the discussion section, because it will be more obvious why we need to assume that $f(0) = f(L) = 0$.

• Let define $\tilde{\psi} : [0, 2\pi] \to \mathbb{C}$ as follows:

$$\tilde{\psi}(x) = f\left(\frac{L}{\pi}x\right).$$

The odd extension of $\tilde{\psi}$, which we are going to denore by $\psi$, is given by

$$\psi(x) = \begin{cases} \tilde{\psi}(x) & x \in [0, \pi] \\ -\tilde{\psi}(-x) & x \in [-\pi, 0] \end{cases}$$

Note that indeed this is an odd function and moreover the the following holds $\psi(0) = \psi(2\pi) = 0$. Also this odd extension we have just created is a continous differentiable function on $[-\pi, \pi]$.

The fact that $\psi \in C'([-\pi,\pi])$ together with the information that $\psi$ is odd (this is how we constructed it), we get that it can be approximated with the following Fourier series:

$$\psi(x) \sim \sum_{n=1}^{\infty} \hat{\psi}_n \sin(nx),$$

with $\hat{\psi}_n$ real numbers.

Note, that

$$\begin{aligned}
\|\psi\|_{L^2(\mathbb{T})}^2 =& \int_{-\pi}^{\pi} \mid \psi(x) \mid^2 \; dx \\
=& 2\int_0^{\pi} \mid \tilde{\psi}(x) \mid^2 \; dx \\
=& 2\int_0^{L} \mid f^2(y)(\frac{\pi}{L}) \mid \; dy \\
=& \frac{2\pi}{L}\|f\|_{L^2(\mathbb{T})}^2,
\end{aligned}$$

and that

$$\begin{aligned}
\|\psi'\|_{L^2(\mathbb{T})}^2 =& \int_{-\pi}^{\pi} \mid \psi'(x) \mid^2 \; dx \\
=& 2\int_0^{\pi} \mid \tilde{\psi}'(x) \mid^2 \; dx \\
=& 2\int_0^{L} \mid (f^2(\frac{xL}{\pi}))'(\frac{L}{\pi})^2 \mid \; dx \\
=& 2(\frac{L}{\pi})^2 \int_0^{L} |f'(y)\frac{\pi}{L}| \, dy \\
=& \frac{2L}{\pi}\|f'\|_{L^2(\mathbb{T})}^2.
\end{aligned}$$

By Perceval's identity , we can see that for $n \geq 1$, we get

$$\|\psi\|_{L^2(\mathbb{T})}^2 = \sum_{n=1}^{\infty} (\hat{\psi}_n)^2 \leq \sum_{n=1}^{\infty} n^2(\hat{\psi}_n)^2 = \|\psi'\|_{L^2(\mathbb{T})}^2,$$

which implies that

$$\|\psi\|_{L^2(\mathbb{T})}^2 \leq \|\psi'\|_{L^2(\mathbb{T})}^2.$$

But this last inequality implies, if we go back to our original inequality, that

$$\frac{2\pi}{L}\|f\|_{L^2([0,1])}^2 \leq \frac{2L}{\pi}\|f'\|_{L^2([0,1])}^2.$$

Simplifying by 2 we get that indeed

$$\|f\|_{L^2([0,1])}^2 \leq \left(\frac{L}{\pi}\right)^2 \|f'\|_{L^2([0,1])}^2.$$

• The inequality is sharp and this can be seen if we manage to find a function for wich we can get an equality. A good and easy example is $f(x) = \sin\left(\frac{x\pi}{L}\right)$ . Obviously $f$ is in $C^1([0,1])$ and satisfies $f(0) = f(1) = 0$. Pluggin in in the inequality we where given to prove we see that indeed we get that

$$\|f\|^2_{L^2([0,1])} = \left(\frac{L}{\pi}\right)^2 \|f'\|^2_{L^2([0,1])}.$$

• To answer to the part where we are asked to show why is it important that $f$ should satisfy $f(0) = f(1) = 0$, we should realize that one major step of the proof was to construct the odd extension, and not just the construction itself was important, but the fact that that construction provided us with a $C^1$ function that has a convergent series.

• To show that you cannot estimate the $L^2$-norm of a smooth, square-integrable function $f : [0,\infty) \to \mathbb{R}$ such that $f(0) = 0$ in terms of the $L^2$ norm of its derivative look at the following example of a smooth, square integrable function :

$$f(x) = e^{-x}x^2 \sin\left(\frac{1}{x^2}\right).$$

Notice that $f(0) = 0$, but the $L^2$-norm of it blows up . □

4. Suppose that $u(x,t)$ is a solution of the following initial value problem for the heat equation

$$u_t = u_{xx} \ x \in \mathbb{T}, t > 0$$
$$u(x,0) = f(x) \ x \in \mathbb{T}$$

where $f \in C(\mathbb{T})$ and

$$u \in C^2(\mathbb{T} \times (0,\infty)) \cap C(\mathbb{T} \times [0,\infty)).$$

(a) Show that

$$u(x,t) = (\theta_t * f)(x) \qquad \text{for } t > 0$$

where

$$\theta_t(x) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{-n^2t} e^{inx}.$$

(b) Show that $u \in C^\infty(\mathbb{T} \times (0,\infty))$.

*Proof.* The solution of this problem can be found in Professor Hunter's book on page 161. Please read it carefully and make sure you understand it. □