201B, Winter '11, Professor John Hunter
Homework 5 Solutions

1. Let $\mathbb{T}^d = \mathbb{T} \times \mathbb{T} \times \cdots \times \mathbb{T}$ denote the d-dimensional Torus.

(a) Show that $\mathcal{B} = \left\{ e^{i\overrightarrow{n}\cdot\overrightarrow{x}} : \overrightarrow{n} \in \mathbb{Z}^d \right\}$ is an orthogonal set in $L^2(\mathbb{T}^d)$ and give an expression for the Fourier coefficients $\hat{f}(\overrightarrow{n})$ of a function

$$f(\overrightarrow{x}) = \sum_{\overrightarrow{n}\in\mathbb{Z}^d} \hat{f}(\overrightarrow{n}) e^{i\overrightarrow{n}\cdot\overrightarrow{x}} \in L^2(\mathbb{T}^d).$$

(You can assume that $\mathcal{B}$ is complete — the proof is similar to the one-dimensional case *e.g.* use an approximate identity

$$\Phi_n(\overrightarrow{x}) = \phi_n(x_1)\phi_n(x_2)\ldots\phi_n(x_d) \qquad n \in \mathbb{N}$$

that is a product of one-dimensional approximate identities $\{\phi_n\}$ consisting of trigonometric polynomials.)

Proof. part (a) from Eric's .tex file
Suppose $\mathbf{n} \neq \mathbf{m}$, then $n_j \neq m_j$ for some $1 \leq j \leq d$ and we have (Fubini theorem allows to change the order of integration)

$$\begin{aligned}
\langle e^{i\mathbf{n}\cdot\mathbf{x}}, e^{i\mathbf{m}\cdot\mathbf{x}} \rangle &= \int_{\mathbb{T}^d} e^{i\mathbf{n}\cdot\mathbf{x}} e^{-i\mathbf{m}\cdot\mathbf{x}} d\curvearrowleft \\
&= \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} e^{in_1x_1} \cdots e^{in_dx_d} e^{-im_1x_1} \cdots e^{-im_dx_d} dx_1 \cdots dx_d \\
&= \int_{\mathbb{T}} e^{in_1x_1} e^{-im_1x_1} \cdots \int_{\mathbb{T}} e^{in_jx_j} e^{-im_jx_j} \cdots \int_{\mathbb{T}} e^{in_dx_d} e^{-im_dx_d} dx_1 \cdots dx_d \\
&= 0.
\end{aligned}$$

So, our Fourier coefficients can be expressed as simply

$$\frac{\langle e^{i\mathbf{n}\cdot\mathbf{x}}, f(\mathbf{x}) \rangle}{(2\pi)^d}$$

□

(b) For $s > 0$, let $H^s(\mathbb{T}^d)$ denote the space of functions $f \in L^2(\mathbb{T}^d)$ such that

$$\sum_{\overrightarrow{n}\in\mathbb{Z}^d} \left(1 + |\overrightarrow{n}|^{2s}\right) \left|\hat{f}(\overrightarrow{n})\right|^2 < \infty.$$

Prove that if $s > d/2$ and $f \in H^s(\mathbb{T}^d)$, then $f \in C(\mathbb{T}^d)$.

Proof. We want to find the conditions that will assure us that the Fourier coefficients of f are absolutely summable. Then using a previous homework, in which we showed that absolute summability implies continuity i.e, if $f \in A(\mathbb{T})$ implies that $f \in C(\mathbb{T}^d)$,

we will be done. .
By the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}^d} \left|\hat{f}(\mathbf{n})\right| &= \sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{1}{(1+|\mathbf{n}|^{2s})^{1/2}}(1+|\mathbf{n}|^{2s})^{1/2}\left|\hat{f}(\mathbf{n})\right|^2 \\ &\leq \left(\sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{1}{(1+|\mathbf{n}|^{2s})}\right)^{1/2}\left(\sum_{\mathbf{n}\in\mathbb{Z}^d}(1+|\mathbf{n}|^{2s})\left|\hat{f}(\mathbf{n})\right|^2\right)^{1/2} \\ &= ||f||_{H^s}\left(\sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{1}{(1+|\mathbf{n}|^{2s})}\right)^{1/2}\end{aligned}$$

We have that $\|f\|_{H^s}$ is bounded since we know that $f \in H^s$. The convergence of the last series can be shown using the Integral Test. Note that we have a radial integral. Let S_d denote the d-dimensional surface area of the unit sphere. Using spherical coordinates (to get an idea how it works, try $d = 2$ or $d = 3$ and then by induction you can deduce the general case) we get:

$$\int_{\mathbb{T}^d} \frac{1}{(1+|\mathbf{x}|^{2s})}\,d\mathbf{x} = S_d \cdot d\int_0^\infty \frac{r^{d-1}}{1+r^{2s}}\,dr.$$

Note: do not forget about multiplying with the Jacobian of the transformation! That is why we got that r^{d-1} under the integral. We conclude that the integral converges for $2s-(d-1) < 1$ or $s > d/2$. □

2. (a) Show that any test function $\phi \in C^\infty(\mathbb{T})$ can be written as $\phi = c + \psi'$ where

$$c = \frac{1}{2\pi}\int \phi\,dx, \qquad \psi \in C^\infty(\mathbb{T}).$$

Proof. Let consider

$$\psi(x) = \int_0^x \phi(t)dt - cx.$$

Then it is easy to check that $\psi(0) = 0$ and $\psi(2\pi) = \int_0^{2\pi} \phi(t)dt - 2\pi c = 0$. Therefore you can see that $\psi \in C^\infty(\mathbb{T})$. Note that by the Fundamental Theorem of Calculus

$$\psi'(x) = \phi(x) - c.$$

Hence, we are done.

□

(b) Suppose that $f \in L^1(\mathbb{T})$ is weakly differentiable and its weak derivative $f' = 0$ is zero. Prove that f = constant (up to pointwise a.e. equivalence).

Proof. For this part I am going to give two solutions, ones that is more general, meaning that it works on more general spaces, and one that works just for this particular problem.

First proof. We begin by writing what mens that $f \in L^1(\mathbb{T})$ is weakly differentiable and its weak derivative $f' = 0$ is zero. this means that for any function $\psi \in C^\infty(\mathbb{T})$ we have

$$\int_{\mathbb{T}} f' \psi \, dx = -\int_{\mathbb{T}} f \psi' \, dx = 0.$$

Hence

$$\int_{\mathbb{T}} f \psi' \, dx = 0.$$

Now, using part (a), we can substitute $\psi'(x)$ by $\phi(x) - c$ we have

$$\begin{aligned} 0 = \int_{\mathbb{T}} f \psi' \, dx &= \int_{\mathbb{T}} f(\phi(x) - c) \, dx \\ &= \int_{\mathbb{T}} f(x)\phi(x) \, dx - c \int_{\mathbb{T}} f(x) \, dx \\ &= \int_{\mathbb{T}} f(x)\phi(x) \, dx - \int_{\mathbb{T}} \phi(y) \, dy \int_{\mathbb{T}} f(x) \, dx. \end{aligned}$$

Rewriting the equation above and denoting $A_f = \displaystyle\int_{\mathbb{T}} f(x) \, dx$, we get

$$\begin{aligned} 0 &= \int_{\mathbb{T}} f(x)\phi(x) \, dx - \int_{\mathbb{T}} \phi(x) \, dx \int_{\mathbb{T}} f(y) \, dy \\ &= \int_{\mathbb{T}} f(x)\phi(x) \, dx - A_f \int_{\mathbb{T}} \phi(x) \, dx \\ &= \int_{\mathbb{T}} \phi(x)(f - A_f) \, dx \end{aligned}$$

The equality above holds for any $\phi \in C^\infty(\mathbb{T})$. therefore we can conclude that $f = A_f$ almost everywhere.

Second proof. *David's proof*

Suppose $n \neq 0$ and let $\phi(x) = -\dfrac{e^{-inx}}{in}$. Then $\phi'(x) = e^{-inx}$ and applying integration by parts shows us that

$$0 = \int_{\mathbb{T}} f'(x)\phi(x) dx = -\int_{\mathbb{T}} f(x)\phi'(x) dx = -\int_{\mathbb{T}} f(x) e^{-inx} dx = -2\pi \hat{f}(n).$$

Thus, $\hat{f}(n) = 0$ for all $n \neq 0$, which necessitates that f is pointwise a.e. equivalent to the constant function $g = \hat{f}(0)$.

$\square$

3. Define the principal-value functional $T : \mathcal{D}(\mathbb{T}) \to \mathbb{C}$ by

$$\begin{aligned}\langle T, \phi\rangle &= \text{p.v.} \int_{\mathbb{T}} \cot\left(\frac{x}{2}\right)\phi(x)\,dx \\ &= \lim_{\epsilon\to 0^+}\left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi}\right)\cot\left(\frac{x}{2}\right)\phi(x)\,dx.\end{aligned}$$

(a) Show that $T \in \mathcal{D}'(\mathbb{T})$ is a well-defined periodic distribution.

Proof. First I will simplify the expression for $\langle T, \phi\rangle$ to

$$\begin{aligned}\langle T, \phi\rangle &= \lim_{\epsilon\to 0^+}\left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi}\right)\cot\left(\frac{x}{2}\right)\phi(x)dx \\ &= \lim_{\epsilon\to 0^+}\left(\int_{\pi}^{\epsilon}\cot\left(\frac{-x}{2}\right)\phi(-x)(-1)\,dx + \int_{\epsilon}^{\pi}\cot\left(\frac{x}{2}\right)\phi(x)\,dx\right) \\ &= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\pi}\cot(\frac{x}{2})\,[\phi(x)-\phi(-x)]\,dx \\ &= \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\pi}\mathbb{E}([\epsilon,\pi])\cot(\frac{x}{2})\,[\phi(x)-\phi(-x)]\,dx.\end{aligned}$$

Here is the point where you will need to switch the limit and the integral, but this can be done just if you have some "nice" properties. Recall that *Lebesgue Monotone Convergence Theorem* takes care of such tricks". Make sure you see why it can be applied here. Having this said we can conclude that

$$\langle T, \phi\rangle \le \int_{0}^{\pi}\cot(\frac{x}{2})\,[\phi(x)-\phi(-x)]\,dx.$$

Observe that linearity is obvious by the linearity of integration. So, we need to show that it is bounded and therefore continuous. We must check the behavior at 0 and π:

$$\begin{aligned}\lim_{x\to 0 \text{ or } \pi}\cot\frac{x}{2}(\phi(x)-\phi(-x)) &= \lim_{x\to 0 \text{ or } \pi}\frac{|\phi(x)-\phi(-x)|}{\tan(\frac{x}{2})} \\ &= 2\lim_{x\to 0 \text{ or } \pi}\frac{\phi'(x)-\phi'(-x)}{\sec^2(\frac{x}{2})} \\ &= 0.\end{aligned}$$

Hence, the integrant is finite at the endpoints, and hence the integral is well defined. Now, let's prove that T is well defined periodic distribution, which means to prove that T is continuous (or bounded if you wish). I prefer to prove the continuity in the old fashion way.

$$\begin{aligned}\langle T, \phi_n\rangle - \langle T, \phi_n\rangle &= |\langle T, \phi_n - \phi\rangle| \\ &\le \int_0^{\pi}\cot(\frac{x}{2})|\phi_n(x)-\phi(x)-(\phi_n(-x)-\phi(-x))|\,dx \\ &\le \int_0^{\pi}\cot(\frac{x}{2})(|\phi_n(x)-\phi(x)|+|\phi_n(-x)-\phi(-x)|)\,dx.\end{aligned}$$

But, we already proven that the RHS is finite, and hence we can use LMCD and move the limit under the integral. We get that T is continous just by looking at the following:

$$\lim_{n\to\infty} \langle T, \phi_n\rangle - \langle T, \phi_n\rangle \leq \int_0^\pi \cot(\frac{x}{2}) \lim_{n\to\infty} (|\phi_n(x) - \phi(x)| + |\phi_n(-x) - \phi(-x)|)\, dx.$$

□

(b) Compute the Fourier coefficients $\hat{T}(n)$ of T.

Proof. Recall that

$$\hat{T}(n) = \frac{1}{2\pi}\langle T, e^{-inx}\rangle.$$

Then,

$$\begin{aligned} 2\pi\hat{T}(n) &= \int_0^\pi \cot\left(\frac{x}{2}\right)\left(e^{-inx} - e^{inx}\right)\, dx \\ &= -2i\int_0^\pi \cot\left(\frac{x}{2}\right)\sin(nx)\, dx \end{aligned}$$

Computing the expression above, we get

$$\hat{T}(n) = -i\int_0^\pi \cot\left(\frac{x}{2}\right)\sin(nx)\, dx.$$

Integrating, you will get:

$$\hat{T}(n) = -i\,\mathrm{sgn}(n).$$

□

4. (a) If $T \in \mathcal{D}'(\mathbb{T})$ is a periodic distribution, show that there exists an integer $k \geq 0$ and a constant C such that

$$|\langle T, \phi\rangle| \leq C\,\|\phi\|_{C^k} \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{T}) \tag{0.1}$$

where

$$\|\phi\|_{C^k} = \sum_{j=0}^{k} \sup_{x\in\mathbb{T}} \left|\phi^{(j)}(x)\right|$$

denotes the C^k-norm of ϕ.

Proof. We will prove the assertion by contradiction. Assume that for all k and C there exists $\phi_{k,C} \in \mathcal{D}$ such that

$$|\langle T, \phi_{k,C}\rangle| > C||\phi_{k,C}||_{C^k}$$

and if we let $k = C$, then we can rename $\phi_{k,C} \equiv \phi_k$. Observe that in this case we have

$$|\langle T, \frac{1}{k}\frac{\phi_k}{||\phi_k||_{C^k}}\rangle| > 1$$

Now, call $\psi_k \equiv \frac{1}{k}\frac{\phi_k}{||\phi_k||_{C^k}}$. For any $j \geq 0$, we have

$$\begin{aligned} ||\psi_k^{(j)}||_\infty &= \frac{1}{k}\frac{||\phi_k^{(j)}||_\infty}{||\phi_k||_{C^k}} \\ &= \frac{1}{k}\frac{||\phi_k^{(j)}||_\infty}{\sum_{i=0}^k \sup_{x\in\mathbb{T}} |\phi^{(i)}(x)|} \\ &\leq \frac{1}{k} \quad \text{for any } k \geq j. \end{aligned}$$

Thus, as $k \to \infty$, $\psi_k \to 0$, but $|\langle T, \psi_k\rangle| > 1$. $\Rightarrow\Leftarrow$ □

□

(b) The order of a distribution T is the minimal integer $k \geq 0$ such that (0.1) holds. What it the order of: (i) a regular distribution; (ii) the delta-function; (iii) the principal value distribution in the previous question? Give an example of a distribution of order 100.

Proof. Three parts:

(i) Let T_f be a *regular distribution* . Then

$$\langle T_f, \phi\rangle = \int_{\mathbb{T}} f\phi dx \leq ||\phi||_\infty \int_{\mathbb{T}} |f| dx = \|\phi\|_{C^0}\|f\|_{L^1}$$

Hence the order of T_f is 0.

(ii) Let T_f be a *delta distribution*. We have

$$\langle \delta, \phi\rangle = \phi(0).$$

Then

$$|\,\langle \delta, \phi\rangle\,| = |\phi(0)| \leq \sup_{x\in\mathbb{T}} |\phi(x)|.$$

So, the delta distribution is 0 order.

(iii) We know that for all $\epsilon > 0$ there is a $\delta > 0$ such that $x \in (0, \delta)$ guarantees $|\cot(x/2)(\phi(x) - \phi(-x))| < 4|\phi'(0)| + \epsilon$.
Therefore,

$$\begin{aligned} |\langle T, \phi\rangle| &= \left|\int_0^\delta \cot\frac{x}{2}(\phi(x) - \phi(-x))dx + \int_\delta^\pi \cot\frac{x}{2}(\phi(x) - \phi(-x))dx\right| \\ &\leq 4|\phi'(0)| + \epsilon + ||\phi||_\infty \int_\delta^\pi \left|\cot\frac{x}{2}\right| dx \\ &\leq C||\phi||_{C^1} \end{aligned}$$

We conclude it has order 1.

(iv) The example, is th following. Consider the 100-th derivative of the delta distribution, which I will denote as $\delta^{(100)}$. Then

$$|\langle \delta^{(100)}, \phi \rangle| = |\langle \delta, \phi^{(100)} \rangle| = |\phi^{(100)}(0)| \leq \sup_{x \in \mathbb{T}} |\phi^{(100)}(0)| \leq \|\phi\|_{C^{100}}.$$

So the 100-th derivative of the delta function has order 100....there are some details that need to be checked in order to conclude that indeed this is the order, and not less. Please let me know if you have trouble proving rigorously that indeed the 100-th derivative of the delta distribution has order 100.

$\square$