201B, Winter '11, Professor John Hunter
Homework 6 Solutions

1. Let X be a (real or complex) linear space and $P, Q : X \to X$ projections.
(a) Show that $I - P$ is the projection onto $\ker P$ along $\operatorname{ran} P$.

Proof. To show that $I - P$ is a projection we need to show that $(I - P)^2 = I - P$. It is easy to see that

$$(I - P)(x) = x - Px,$$

and therefore

$$\begin{aligned}(I - P)^2(x) =&(I - P)(x - Px)\\ =&I(x - Px) - P(x - Px)\\ =&x - Px - Px + P^2(x)\\ =&x - Px - Px + Px\\ =&x - Px\\ =&(I - P)(x).\end{aligned}$$

Note that we used that P is a projection, which by definition means in particular that $P^2 = P$. We still have to prove $I - P$ is the projection onto $\ker P$ along $\operatorname{ran} P$.
• Let $x \in \ker P$. Then $Px = 0$. So

$$(I - P)x = x - Px = x.$$

• Let $y \in \operatorname{ran}(I - P)$. Then $y = (I - P)v = v - Pv$ for some $v \in X$. So

$$Py = P(v - Pv) = 0.$$

Therefore, $\operatorname{ran}(I - P) = \ker P$.
• Similarly, we can show that $\operatorname{ran} P = \ker(I - P)$.

□

(b) The projections P, Q are orthogonal, written $P \perp Q$, if $PQ = QP = 0$. Show that $P + Q$ is a projection if and only if $P \perp Q$.

Proof. First let's assume that $P \perp Q$. We want to prove that $P + Q$ is a projection and for this we will use that $P^2 = P$ and that $Q^2 = Q$:

$$\begin{aligned}(P + Q)^2(x) =&(P + Q)(Px + Qx)\\ =&P(Px + Qx) + Q(Px + Qx)\\ =&P^2(x) + PQ(x) + QP(x) + Q^2(x)\\ =&P(x) + Q(x).\end{aligned}$$

Now, let's assume we know that $P + Q$ is a projection and we want to prove that this implies that $P \perp Q$. Hence, since we assumed that $P + Q$ is a projection , we know that this equality holds $(P+Q)^2 = P+Q$, i.e.,

$$\begin{aligned}(P+Q)^2(x) =&(P+Q)(Px+Qx)\\ =&P(Px+Qx)+Q(Px+Qx)\\ =&P^2(x)+PQ(x)+QP(x)+Q^2(x)\\ =&P(x)+PQ(x)+QP(x)+Q(x)\\ =&(P+Q)(x)+PQ(x)+QP(x).\end{aligned}$$

This means that

$$0 = PQ(x)+QP(x).$$

Hence, $PQ = -QP$.

Let $x \in \operatorname{ran} PQ$. We have that

$$PQx = x.$$

Applying P to both sides of this equation, we get $Px = x$.
Analougus,

$$-QPx = x.$$

Applying Q to both sides gives us $Qx = x$.
Therefore

$$x = -QPx = -Qx = -x.$$

Hence, $x = 0$, which implies $PQ = 0 = QP$.

□

(c) If the projections P, Q commute, show that PQ is the projection onto $\operatorname{ran} P \cap \operatorname{ran} Q$ along $\ker P + \ker Q$.

Proof. We know that $PQ = QP$. To shwo that PQ is the projection onto $\operatorname{ran} P \cap \operatorname{ran} Q$ along $\ker P + \ker Q$, let's apply the definition:

$$\begin{aligned}(PQ)^2(x) =&(PQ)(P(Qx))\\ =&(PQ)(Q(Px))\\ =&P(Q^2(Px))\\ =&P(Q(Px))\\ =&P(P(Qx))\\ =&P^2(Qx)\\ =&(PQ)(x).\end{aligned}$$

Hence, we proved that PQ is the projection. Remains to show that is a projection onto $\operatorname{ran} P \cap \operatorname{ran} Q$ along $\ker P + \ker Q$.

- Let $x \in \operatorname{ran} P \cap \operatorname{ran} Q$. Then $Px = x = Qx$. So $PQx = Px = x$ i.e., $x \in \operatorname{ran} PQ$.

• Let $y \in \operatorname{ran} PQ$. Then $PQy = y = QPy$. So, applying the samse stratergy as in part (b) we see that $y \in \operatorname{ran} P$ and $y \in \operatorname{ran} Q$.

Therefore, $\operatorname{ran} PQ = (\operatorname{ran} P \cap \operatorname{ran} Q)$.

• Let $x \in \ker P + \ker Q$ and $y \in \ker PQ$. Writing x as $x = u + v$ where $u \in \ker P$ and $v \in \ker Q$ we have

$$PQx = PQ(u + v) = PQu + PQv = PQu = QPu = 0.$$

So $\ker P + \ker Q \subset \ker PQ$.

• Let $x \in \ker PQ$. Then, either $x \in \ker Q$ or $x \in (\ker Q)^C$. If $x \in \ker Q$, then $x = 0 + x$. If $x \in (\ker Q)^C$, then $Qx \in \ker P$.

Hence x can be expressed as $x = x + Qx - Qx$. Observe that $x - Qx \in \ker Q$.

Therefore, $\ker PQ = \ker P + \ker Q$.

□

(d) Give an example (or examples) to show that $P + Q$ need not be a projection if $PQ = 0$ but $QP \neq 0$, and PQ need not be a projection if P,Q do not commute.

Proof. Consider the prijection given by the matrices as follows:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Please check that these matrices are projections, but their sums are not. □

2. Let $\mathcal{H} = L^2(\mathbb{R})$. For any Lebesgue measurable set $A \subset \mathbb{R}$, define

$$P_A : \mathcal{H} \to \mathcal{H}$$

by $P_A f = \chi_A f$ where χ_A is the characteristic function of A. (We define $P_\emptyset = 0$.) Show that P_A is an orthogonal projection. What are its range and kernel? Show that P_A, P_B commute. What is $P_A P_B$? When is $P_A \perp P_B$? What is $P_A + P_B$ in that case?

Proof. • $P_A f = \chi_A f$ where χ_A is the characteristic function of A. From the following computations:

$$P_A(P_A f) = P_A(\chi_A f) = \chi_A(P_A f) = \chi_A(\chi_A f) = \chi_A f = P_A f,$$

we get that $P_A f$ is a projection on $\mathcal{H}$. To prove that indeed is a orthogonal projection we need to show that

$$\langle P_A f, g \rangle = \langle P_A g, f \rangle \text{ for any } f,\, g \in L^2(\mathbb{R}).$$

But this translates to:

$$\begin{aligned}\langle P_A f, g\rangle &= \int_{\mathbb{R}} \chi_A f \overline{g}\, dx \\ &= \int_{\mathbb{R}} \chi_A f\, g\, dx \\ &= \int_{\mathbb{R}} \chi_A g \overline{f}\, dx \\ &= \int_{\mathbb{R}} \chi_A g f\, dx \\ &= \langle P_A g, f\rangle\,.\end{aligned}$$

Hence we proved that $P_A f$ is a orthogonal projection on $\mathcal{H}$.

- $\mathrm{Ker}(P_A) = \{f \in L^2(\mathbb{R}) \mid P_A f = 0\}$. Hence

$$\ker P_A = \left\{f \in L^2(\mathbb{R}) : f = 0 \text{ a.e. on } A\right\}$$

- From the above rationament we conclude that the ran $P_A = \left\{f \in L^2(\mathbb{R}) : f = 0 \text{ a.e. on } A^C\right\}$

.

- Let A and $B \subset \mathbb{R}$, then $P_A f = \chi_A f$ where χ_A is the characteristic function of A, and $P_B f = \chi_B f$ where χ_B is the characteristic function of B. For any $f \in \mathcal{H}$

$$\begin{aligned}(P_A P_B)(f) =& P_A(P_B(f)) \\ =& P_A(\chi_B f) \\ =& \chi_A \chi_B f \\ =& \chi_B \chi_A f \\ =& P_B(\chi_A f) \\ =& P_B(P_A(f)) \\ =& (P_B P_A)(f).\end{aligned}$$

Hence, P_A and P_B commute.

- Looking at the expression of $P_A P_B$ we've wrote above , we see that

$$(P_A P_B)(f) = \chi_B \chi_A f = \chi_{A\cap B} f$$

- When $P_A \perp P_B$?

  Answer: $P_A \perp P_B$ if $A \cap B = \emptyset$.

- In this case, i.e., when $P_A \perp P_B$ then $P_A + P_B$?

  Answer:$P_A + P_B = P_{A\cup B}$.

$\square$

3. Suppose that $\mathcal{H}$ is a separable Hilbert space with ON basis $\{e_n : n \in \mathbb{N}\}$. Let M be the closed linear span of

$$e_1, \quad e_3, \quad e_5, \quad e_7, \quad \ldots$$

and N the closed linear span of

$$e_1 + \frac{1}{2}e_2, \quad e_3 + \frac{1}{2^2}e_4, \quad e_5 + \frac{1}{2^3}e_6, \quad e_7 + \frac{1}{2^3}e_8 \quad \ldots.$$

(a) Show that $M \cap N = \{0\}$. If $X = M \oplus N$, show that

$$\overline{X} = \mathcal{H}, \qquad X \neq \mathcal{H}.$$

(Thus, X is an inner-product space when equipped with the $\mathcal{H}$-inner-product.)

Proof. • Suppose $x \in M \cap N$ and let's take $x \neq 0$ because it is obviously that $0 \in M \cap N$. Then x may be expressed in terms of the elements of the basis of M as follows

$$x = \sum_{i \in \mathbb{N}} x_i e_i$$

and also in terms of the elements of the basis N as follows

$$x = \sum_{i \in \mathbb{N}} \tilde{x}_i \left(e_i + \frac{1}{2^{\frac{i+1}{2}}} e_{i+1} \right).$$

There must be an i, which I will denote by $j \in \mathbb{N}$, such that $x_j \neq 0$. In this case, we must have $\tilde{x}_j \neq 0$, which in particular means that the coefficient of e_{j+1} is nonzero. But e_{j+1} is an element in $N \setminus M$. Therefore, $M \cap N = \{0\}$.

• Now, I want to prove that given $X = M \oplus N$, then $\overline{X} = \mathcal{H}$ and also $X \neq \mathcal{H}$. We know $\mathcal{H}$ is a Hilbert space, so in particular it is complete; this implies that $X \subset \mathcal{H}$. Hence, we need to prove that there exists an element m that belongs to $\mathcal{H} \setminus X$. Since $\{e_n \mid n \in \mathbb{N}\}$ is an ON basis for $\mathcal{H}$, then we can write m in terms of the elements of the basis:

$$m = \sum_{i \in \mathbb{N}} x_i e_i.$$

But also we can express the element m in terms of the basis of N and also in terms of the basis of M. Matching i=even (which are the terms in N) and j=odd (which are the terms in M) we get:

$$x_i 2^{\frac{i}{2}} \left(e_{i-1} + \frac{1}{2^{\frac{i}{2}}} e_i \right) = x_i 2^{\frac{i}{2}} e_i + x_i e_i \text{ in } N$$
$$x_i e_i - x_{i+1} 2^{\frac{i+1}{2}} e_i = \left(x_i - x_{i+1} 2^{\frac{i+1}{2}} \right) e_i \text{ in } M$$

.

Having this said we can "approximate" m by the following sequence of elements:

$$m_n = \sum_{i = even \in \mathbb{N}}^{n} x_i 2^{\frac{i}{2}} \left(e_{i-1} + \frac{1}{2^{\frac{i}{2}}} e_i \right) + \sum_{i = odd \in \mathbb{N}}^{n} \left(x_i - x_{i+1} 2^{\frac{i+1}{2}} \right) e_i.$$

Note that the RHS has two terms, and it is trivial to see that the first term belongs to N and the second term belongs to M. To be more clear the reason those partial sums are in N, respectively in M, is because partial sums with finite terms are always convergent. Take the limit as n goes to ∞ from m_n and set $m = \lim_{n\to\infty}$ and also set $x_i = \frac{1}{i}$. We can see that using Parceval's Theorem, we get that

$$\sum_{n}^{\infty} \frac{e_i}{i} = m.$$

But, $m \in \mathcal{H}$ because the series $\sum_{n}^{\infty} \frac{1}{i^2}$ is convergent. Therefore, $m \notin M \oplus N$. This is true because of the partial sum of the terms of N,which made us observe that

$$\sum_{i=odd\,\in\mathbb{N}} \frac{2^i}{i}e_i + \frac{1}{i}e_i$$

doesn't converge since by the n-th therm test we can see that $\dfrac{2^{\frac{i}{2}}}{i}$ goes to ∞ as i goes to ∞.

□

(b) Let $P : X \to X$ be the projection of X onto M along N. Show that P is unbounded.

Proof. Let's compute the norm of P where

$$\|P\| = \sup_{\|x\|=1} \|Px\|.$$

Take the sequence $x_n := e_{2n}$ and see how P is acting on the elements of this sequence.
Note that the norm of x_n is 1, i.e., $\|x_n\| = 1$. We have the following ways of writing the elements of x_n sequence . The idea is to write them in terms of the basis of M and N respectively;

$$e_2 = 2(e_1 + \frac{1}{2}e_2) - 2e_1.$$

Therefore

$$Px_1 = Pe_2 = \|2e_1\| = 2.$$

Continuing we get

$$e_{2n} = 2^n(e_{2n-1} + \frac{1}{2^n}e_{2n}) - 2^n e_{2n-1}.$$

Therefore

$$Px_n = Pe_{2n} = \|2e_{2n-1}\| = 2^n.$$

Hence, as $n \to \infty$, $\|Px_n\| \to \infty$. So P is unbounded. □

4. Let $\mathcal{H} = H^1(\mathbb{T})$ denote the Sobolev space of 2π-periodic functions in $L^2(\mathbb{T})$ whose weak derivative belongs to $L^2(\mathbb{T})$ with inner product

$$\langle u, v\rangle_{\mathcal{H}} = \int_{\mathbb{T}} (\overline{u}v + \overline{u}'v')\, dx.$$

For $f \in L^2(\mathbb{T})$, define $F : \mathcal{H} \to \mathbb{C}$ by

$$F(v) = \int_{\mathbb{T}} \overline{f} v\, dx.$$

Show that $F \in \mathcal{H}^*$ and find the element $u \in \mathcal{H}$ such that

$$F(v) = \langle u, v\rangle_{\mathcal{H}}.$$

What is $\|F\|_{\mathcal{H}^*}$?

Proof. To show that $F \in \mathcal{H}^*$ we need to show that the functional is bounded in $H^1(\mathbb{T})$:

$$\begin{aligned}|F(v)| =& |\int_{\mathbb{T}} \overline{f} v|\, dx \\ \leq& \int_{\mathbb{T}} |\overline{f}||v|\, dx\end{aligned}$$

Applying Cauchy-Schwartz inequality, we get

$$\begin{aligned}|F(v)| \leq& \int_{\mathbb{T}} |\overline{f}||v|\, dx \\ \leq& \left(\int_{\mathbb{T}} |\overline{f}|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}} |v|^2\, dx\right)^{\frac{1}{2}} \\ =& \|f\|_{L^2(\mathbb{T})}\|v\|_{L^2(\mathbb{T})} \\ \leq& \|f\|_{L^2(\mathbb{T})}\|v\|_{H^1(\mathbb{T})}.\end{aligned}$$

By Riesz representation theorem, since F is a bounded linear functional on the Hilbert space $\mathcal{H}$, then there is a unique vector $u \in \mathcal{H}$ such that

$$F(v) = \langle u, v\rangle_{\mathcal{H}}.$$

Let's find u such that

$$\int_{\mathbb{T}} \overline{f} v\, dx = \int_{\mathbb{T}} (\overline{u}v + \overline{u}'v')\, dx.$$

We can rewrite the expression above as follows:

$$\int_{\mathbb{T}} \left(\overline{f}v - \overline{u}v - \overline{u}'v'\right)\, dx = 0,$$

which is equivalent to (integration by parts):

$$\begin{aligned}\int_{\mathbb{T}} \left(\overline{f}v - \overline{u}v + \overline{u}''v\right)\, dx = 0,\ \text{for } \forall v \in \mathcal{H} \\ \int_{\mathbb{T}} \left(\overline{f} - \overline{u} + \overline{u}''\right) v\, dx = 0,\ \text{for } \forall v \in \mathcal{H}.\end{aligned}$$

But this means that u is a weak solution of the ODE:

$$\overline{f} - \overline{u} + \overline{u}'' = 0.$$

Rearranging the terms, and taking the complex conjugate , we get that the ODE can be written as:

$$u'' - u = -f,$$

where $f \in L^2(\mathbb{T})$. This is a second order inhomogeneous constant coefficient ODE, which can be solved as follows:
1) You solve the homogeneous ODE: $u'' - u = 0$ and find the homogeneous solution u_h
2) You look for a particular solution u_p which can be found using the *variation of parameters*,
3) The solution is given by $u = u_h + u_p$. This way is much harder, in the sense that the computations get messier than expected. I have managed to finish them, but doesn't deserve the time to type them up. So better let's try something easier.

Easier way to compute the norm!

We will solve that ODE writing u by its Fourier series. Since $u \in H^1(\mathbb{T})$, then f, $u^{'}$, and $u \in L^2(\mathbb{T})$.
Hence,

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx},$$
$$u = \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{inx},$$
$$u = -\sum_{n\in\mathbb{Z}} n^2\hat{u}(n)e^{inx}.$$

Plugging in in $u'' - u = -f$ we get:

$$-\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx} = -\sum_{n\in\mathbb{Z}} n^2\hat{u}(n)e^{inx} - \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{inx}.$$

This is equivalent to

$$\hat{f}(n) = \hat{u}(n) + n^2\hat{u}(n)$$

or

$$\hat{u}(n) = \frac{\hat{f}(n)}{1+n^2}.$$

Thus,

$$u = \sum_{n\in\mathbb{Z}} \frac{\hat{f}(n)}{1+n^2}e^{inx}.$$

Note that we didn't really know much about $u^{''}$, meaning we didn't know its regularity, but we can prove that it belongs to $L^2(\mathbb{T})$, by showing that its Fourier coefficients are square summable. This is easy to see:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\hat{u}^{''}(n)|^2 =& \sum_{n\in\mathbb{Z}} |\frac{n^2\hat{f}(n)}{1+n^2}|^2 \\ \leq& \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 \\ \leq& \infty,\end{aligned}$$

because $f \in L^2(\mathbb{T})$. Therefore $u^{''} \in L^2(\mathbb{T})$

Now, applying the Riesz representation theorem and Parseval's identity, we see that

$$\begin{aligned}\|F\|_{\mathcal{H}^*} =& \|u\|_{\mathcal{H}} \\ =& |\langle u, u\rangle_{\mathcal{H}}|^{\frac{1}{2}} \\ =& \left(\int_{\mathbb{T}} (\overline{u}u + \overline{u^{'}}u^{'})\, dx\right)^{\frac{1}{2}} \\ =& \left(2\pi\sum_{n\in\mathbb{Z}} |\hat{u}(n)|^2 + 2\pi\sum_{n\in\mathbb{Z}} |\hat{u}^{'}(n)|^2\right)^{\frac{1}{2}} \\ =& \left(2\pi\sum_{n\in\mathbb{Z}} |\frac{\hat{f}(n)}{1+n^2}|^2 + 2\pi\sum_{n\in\mathbb{Z}} |\frac{n\hat{f}(n)}{1+n^2}|^2\right)^{\frac{1}{2}} \\ =& \left(2\pi\sum_{n\in\mathbb{Z}} \frac{(1+n^2)|\hat{f}(n)|^2}{(1+n^2)^2}|^2\right)^{\frac{1}{2}} \\ =& \sqrt{2\pi}\left(\sum_{n\in\mathbb{Z}} \frac{|\hat{f}(n)|^2}{(1+n^2)}\right)^{\frac{1}{2}}.\end{aligned}$$

We are done! □