201B, Winter '11, Professor John Hunter
Homework 7 Solutions

1. Let $\mathcal{H} = L^2(0,1)$ with the standard inner product

$$\langle f, g\rangle = \int_0^1 \overline{f}(x)g(x)\,dx.$$

Define $M : \mathcal{H} \to \mathcal{H}$ by

$$(Mf)(x) = xf(x)$$

i.e. M is multiplication by x.

(a) Show that M is a bounded self-adjoint linear operator on $\mathcal{H}$ and find $\|M\|$.

• The boundedness of the operator M follows from:

$$\begin{aligned}
\|Mf\|_{\mathcal{H}} =& \left(\int_0^1 \mid Mf \mid^2\right)^{\frac{1}{2}} \\
=& \left(\int_0^1 \mid xf(x) \mid^2\right)^{\frac{1}{2}} \\
\leq& \sup_{x\in[0,1]} \mid x \mid \left(\int_0^1 \mid f(x) \mid^2\right)^{\frac{1}{2}} \\
\leq& \sup_{x\in[0,1]} \mid x \mid \left(\int_0^1 \mid f(x) \mid^2\right)^{\frac{1}{2}} \\
\leq& \left(\int_0^1 \mid f(x) \mid^2\right)^{\frac{1}{2}} \\
=& \|f\|_{\mathcal{H}}
\end{aligned}$$

Note, that here we used the fact that $\|x\|_\infty$ is finite since x is a continuous function on a compact set, and therefore it is bounded. It follows that

$$\|Mf\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}$$

i.e.,

$$\|M\|_{\mathcal{H}} \leq 1$$

• The linearity of M is trivial to show.

• M is self-adjoint because of the following

$$\begin{aligned}\langle Mf, g\rangle &= \int_0^1 x\overline{f(x)}g(x)\,dx \\ &= \int_0^1 \overline{f(x)}\overline{\overline{xg(x)}}\,dx \\ &= \langle f, Mg\rangle\,.\end{aligned}$$

Note that here we used the fact that $x = \overline{x}$ since x is a real-valued function.

• We want to conclude that the norm of the operator M is indeed 1. For this consider the sequence of functions $f_k(x) = \sqrt{2k+1}x^k$.

We have

$$\int_0^1 (f_k(x))^2\,dx = \frac{2k+1}{2k+1}x\mid_0^1 = 1,$$

for all $k \in \mathbb{N}$.

Computing the norm of M evaluated at f_k we get

$$\|Mf_k\| = \int_0^1 (2k+1)x^{2k+2}\,dx = \frac{2k+1}{2k+3}.$$

This approaches 1 as k goes to infinity, and hence we are done showing that the norm of M is indeed equal to 1.

$\square$

(b) What is the kernel of M? What is the range of M? Is M onto? Is $\operatorname{ran} M$ closed?

• **Kernel:** Note for $xf = 0$ we must have that $f = 0$ a.e., so M is injective i.e., the kernel of our operator is trivial (means it contains just the 0 vector).

• **Surjectivity:** M is not onto since for $Mf(x) = xf(x) = g(x) \in L^2([0,1])$, we must have $f(x) = \frac{g(x)}{x}$. Let's for example pick $g(x)$ to be the constant function $g(x) = 1$, then we end up getting that $f(x) = \frac{1}{x}$, which is not an $L^2(0,1)$ function, as we wanted. Therefore, the range of $M = \left\{f \in L^2([0,1)) : \frac{f}{x} \in L^2([0,1])\right\}$.

The fact that range of M is not closed comes from: choose a sequence of functions defined by

$$f_n(x) = \chi_{[\frac{1}{n},1]}\frac{1}{x}$$

which obviously belongs to $L^2(0,1)$, but it is obvious that $Mf_n \to 1 \notin$ Range of M.

Another way to see that indeed the range of M is not closed, is to use the fact that M is self-adjoint, and by a theorem studied in class we get

$$\mathcal{H} = \overline{ran\, M} \oplus ker\, M,$$

but since $ker\, M = \{\, 0\}$ and since $ran\, M \neq \mathcal{H}$, we get that $ran\, M \neq \overline{ran\, M}$.

2. Let $\mathcal{H}$ be a complex Hilbert space.

(a) If $A, B \in \mathcal{B}(\mathcal{H})$ are bounded linear operators on $\mathcal{H}$ such that

$$\langle x, Ax\rangle = \langle x, Bx\rangle \qquad \text{for all } x \in \mathcal{H}$$

show that $A = B$.

Proof. Proof. Suppose $\langle x, Ax\rangle = \langle x, Bx\rangle$. Let $C = A - B$, then

$$\langle x, Cx\rangle = 0, \forall x \in \mathcal{H}.$$

Hence for any $x, y \in \mathcal{H}$,

$$\begin{aligned} 0 = \langle x+y, C(x+y)\rangle &= \langle x, Cx\rangle + \langle y, Cy\rangle + \langle x, Cy\rangle + \langle y, Cx\rangle \\ &= \langle x, Cy\rangle + \langle y, Cx\rangle. \end{aligned}$$

Similarly, we can prove $\langle x, Cy\rangle - \langle y, Cx\rangle = 0$ by considering $\langle x+iy, C(x+iy)\rangle$. Then we find

$$\langle x, Cy\rangle = 0, \quad \forall x, y \in \mathcal{H}$$

which implies $C = 0$. Therefore $A = B$.

□

(b) Show that an operator $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint if and only if $\langle x, Ax\rangle$ is real for all $x \in \mathcal{H}$.

Proof. • Suppose that $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint. Then by definition we have:

$$\langle x, Ax\rangle = \overline{\langle Ax, x\rangle} = \overline{\langle x, Ax\rangle},$$

for all $x \in \mathcal{H}$. Note that this implies that $\langle x, Ax\rangle \in \mathbb{R}$ (a number equals its complex conjugate just when that number belongs to $\mathbb{R}$).

• Now let's prove the second implication. Suppose that $\langle x, Ax\rangle \in \mathbb{R}$ then

$$\langle x, Ax\rangle = \langle A^*x, x\rangle = \overline{\langle x, A^*\rangle} = \langle x, A^*x\rangle\,.$$

From probelm no 2, part a), we just solved above, we indeed get $A = A^*$.

□

(c) Do these results remain true if $\mathcal{H}$ is a real Hilbert space?

Proof. Due to Tim.
No, the results do not remain true. To see that a) fails in a real Hilbert space, consider the operators A and B on $\mathbb{R}^2$ given by rotation by $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, respectively. Then $\langle x, Ax\rangle = 0 = \langle x, Bx\rangle$, but these operators are clearly not identical.

To see that b) fails, consider the left-shift operator on $\ell^2(\mathbb{N})$ over $\mathbb{R}$. Then the inner product will be real, but this operator is not self-adjoint.

$\square$

3. Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded, self-adjoint linear operator such that there is a constant $c > 0$ with

$$c\|x\| \leq \|Ax\| \qquad \text{for all } x \in \mathcal{H}.$$

Prove that there is a unique solution x of the equation $Ax = y$ for every $y \in \mathcal{H}$.

Proof. Note that the problem is asking you to prove that in the conditions mentioned above, A has an inverse, which is a bounded operator. This problem can be solved in at least two ways, but here is one of the proofs.

If $x \in \ker(A)$, then $Ax = 0$. Since $c\|x\| \leq \|Ax\| = 0$, we find $x = 0$. Therefore A is an injection.
Since A is self-adjoint, we find

$$\operatorname{ran}(A)^{\perp} = \ker(A) = \{0\}. \tag{0.1}$$

Hence $\overline{\operatorname{ran}(A)} = \mathcal{H}$. If $y \in \mathcal{H}$ and (x_n) is a sequence in $\mathcal{H}$ so that

$$\lim_{n\to\infty} \|Ax_n - y\| = 0. \tag{0.2}$$

We find (Ax_n) is a Cauchy sequence in $\mathcal{H}$. Since

$$c\|x_n - x_m\| \leq \|A(x_n - x_m)\| = \|Ax_n - Ax_m\|, \tag{0.3}$$

we find (x_n) is also a Cauchy sequence in $\mathcal{H}$. Since $\mathcal{H}$ is a Hilbert space, we can find $x \in \mathcal{H}$ so that $\lim_{n\to\infty} \|x_n - x\| = 0$. Since A is a bounded operator, we find $y = \lim_{n\to\infty} Ax_n = Ax$.
Thus $y \in \operatorname{ran}(A)$. Hence $\operatorname{ran}(A) = \mathcal{H}$. This proves that A is a bounded open bijection. The open mapping theorem implies that A is invertible. Therefore for each $y \in \mathcal{H}$, $Ax = y$ has a unique solution. $\square$

4. A Laurent operator (or discrete convolution) is a bounded linear operator A on $\ell^2(\mathbb{Z})$ whose matrix with respect to the standard basis is

$$[A] = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_0 & a_{-1} & a_{-2} & \cdot \\ \cdot & a_1 & a_0 & a_{-1} & \cdot \\ \cdot & a_2 & a_1 & a_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where $a_n \in \mathbb{C}$, meaning that

$$(Ax)_m = \sum_{n=-\infty}^{\infty} a_{m-n} x_n.$$

(a) Let $S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ denote the right shift operator, defined by

$$(Sx)_m = x_{m-1}.$$

Show that a bounded linear operator on $\ell^2(\mathbb{Z})$ is a Laurent operator if and only if it commutes with S.

Proof. • Suppose A is a Laurent operator. First implication I am going to prove is to assume A is a Laurent operator and I want to show that it commutes with S.
Then

$$A(Sx)_m = (Ax)_{m-1} = \sum_{n \in \mathbb{Z}} a_{m-n-1} x_n.$$

Also,

$$\begin{aligned} S(Ax)_m =& S\left(\sum_{n \in \mathbb{Z}} a_{m-n} x_n\right) \\ =& \sum_{n \in \mathbb{Z}} S(a_{m-n} x_n) \\ =& \sum_{n \in \mathbb{Z}} a_{m-n} x_{n-1} \\ =& \sum_{k \in \mathbb{Z}} a_{m-k-1} x_k. \end{aligned}$$

Hence, S and A commute.

• Now, let $A \in \mathcal{B}(\ell^2(\mathbb{Z}))$. The goal now is to show that A is a Laurent operator, knowing that it commutes with S.
We can represent A as a matrix of the form as follows

$$A\mathbf{x} = (a_{m,n})(x_n) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} a_{m,n} x_n.$$

As I said, we are in the case where we suppose that A and S commute.

Then,

$$\begin{aligned}AS\mathbf{x} =& \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a_{m,n}x_{n-1}\\ =& SA\mathbf{x}\\ =& S\left(\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a_{m,n}x_n\right)\\ =& \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a_{m-1,n-1}x_{n-1}.\end{aligned}$$

Hence, we got that $a_{m,n} = a_{m-1,n-1}$. Therefore, A is a Laurent operator. □

(b) Let $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ denote the unitary Fourier transform

$$\mathcal{F}f = \hat{f} \qquad \hat{f}(n) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}} f(x)e^{-inx}\,dx.$$

Suppose that $M : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the bounded multiplication operator

$$(Mf)(x) = a(x)f(x)$$

corresponding to multiplication by a function $a \in L^\infty(\mathbb{T})$. Show that

$$A = \mathcal{F}M\mathcal{F}^{-1}$$

is a Laurent operator whose matrix entries are the Fourier coefficients of a. What function $s(x)$ corresponds to S?

Proof. The idea is that when you want to determine an operator, it is enough to determine how the operator, which in our case is A, acts on the basis of the space on which was defined; in our case on the basis $\{e_n\}$ of $\ell^2(\mathbb{Z})$.
To determine the entries of the matrix, we need to see how the matrix acts on the basis mentioned above. Note that the nth basis vector e_n, is in fact the Fourier coefficient of e^{inx}.

Then

$$A(e_n) = \mathcal{F}M\mathcal{F}^{-1}(e_n) = \mathcal{F}M(e^{inx}) = \mathcal{F}(a(x)e^{inx}),$$

which implies that

$$\widehat{a(x)e^{inx}}_k = \int_{\mathbb{T}} a(x)e^{inx}e^{-ikx}\,dx = \int_{\mathbb{T}} a(x)e^{i(n-k)x}\,dx = \hat{a}_{k-n}.$$

Since any sequence $b \in l^2(\mathbb{Z})$ can be written as a linear combination of the elements of the basis, i.e., $\mathbf{b} = \sum_{n\in\mathbb{Z}} b_n e_n$, we have:

$$(A\mathbf{b})_k = A\left(\sum_{n\in\mathbb{Z}} b_n e_n\right) = \sum_{n\in\mathbb{Z}} b_n A(e_n) = \sum_{n\in\mathbb{Z}} b_n \hat{a}_{k-n},$$

which is of the form of a Laurent operator.

Note that the matrix representation for S is

$$[S] = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 0 & 0 & \cdot \\ \cdot & 1 & 0 & 0 & \cdot \\ \cdot & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Therefore we need a function f that has the only nonzero Fourier coefficient is $\hat{f}(1) = 1$. Hence S corresponds to multiplication by $s(x) = e^{ix}$. $\square$

(c) Deduce that if a is nonzero, except possibly on a set of measure zero, and $1/a \in L^\infty(\mathbb{T})$, then the corresponding Laurent operator A is invertible. If

$$\left[A^{-1}\right] = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & b_0 & b_{-1} & b_{-2} & \cdot \\ \cdot & b_1 & b_0 & b_{-1} & \cdot \\ \cdot & b_2 & b_1 & b_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

give an expression for the coefficients b_n in terms of a.

Proof. From Tim's .tex file Note that if $a(x)$ is non-zero off of a set of measure zero and $\frac{1}{a(x)} \in L^\infty(\mathbb{T})$, then the operator $N : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ define by $Nf = \frac{f(x)}{a(x)}$ is bounded and linear.

Now, consider the operator $B = \mathcal{F}N\mathcal{F}^{-1}$.

Then,

$$\begin{aligned} AB =& \mathcal{F}M\mathcal{F}^{-1}\mathcal{F}N\mathcal{F}^{-1} \\ =& \mathcal{F}MN\mathcal{F}^{-1} \\ =& \mathcal{F}\mathcal{F}^{-1} \\ =& I. \end{aligned}$$

So, by part (b), the matrix entries of A^{-1} will be given by the Fourier coefficients of $\dfrac{1}{a(x)}$.

$\square$