201B, Winter '11, Professor John Hunter
Homework 8 Solutions

1. A sequence of bounded linear operators $A_n \in \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is said to converge to an operator $A \in \mathcal{H}$: *uniformly* if $A_n \to A$ with respect to the operator norm on $\mathcal{B}(\mathcal{H})$; *strongly* if $A_n x \to Ax$ strongly in $\mathcal{H}$ for every $x \in \mathcal{H}$.

   (a) Give an example of a sequence of operators that converges strongly but not uniformly.

   Proof. Remember we did this in 201A. Solution due to Eric!

   • Let $T_n = T^n$ where T is the left shift operator on $\ell^2(\mathbb{Z})$ given by

   $$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

   First, we will show that T_n converges strongly to 0. Then we will show that it doesn't converge uniformly to zero.
   Pick any sequence $x = (x_1, x_2, \ldots)$ in $\ell^2(\mathbb{Z})$; we have that

   $$x_n \to 0 \text{ as } n \to \infty.$$

   Remember that the $\ell^2(\mathbb{Z})$-norm of $T_n x$ is given as follows:

   $$||T_n x|| = \left(\sum_{i=n}^{\infty} x_n^2\right)^{1/2} < \infty,$$

   which monotonically approaches 0 as $n \to \infty$.
   So, T_n converges strongly to the zero operator.

   • If T_n converged uniformly, it would have to agree with the strong limit we have found above i.e., it suppose to be 0. We can calculate the norm of $||T_n||$ by first noting that clearly $||T_n|| \leq 1$. To prove that indeed the norm of T_n is 1, we do the usual trick: if we take any sequence $s_n \in \ell^2(\mathbb{T})$ that begins with n zeros then $||T_n s_n|| = ||s_n||$, which implies that $||T_n|| \geq 1$. Thus, $||T_n|| = 1$ for all $n \in \mathbb{N}$.

   Therefore we can conclude that, T_n does not converge uniformly since we cannot have

   $$||T_n|| = 1 \to 0.$$

   $\square$

   (b) Give an example of a sequence of operators that converges weakly but not strongly.

Proof. Remember we did this in 201A. Solution due to Eric!

• Let $S_n = S^n$ where S is the right shift operator on $\ell^2(\mathbb{Z})$ given by

$$S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots).$$

Our proof here will mirror the structure of the argument in part (a). First, we show that S_n converges weakly to zero. Secondly, we will show that it cannot converge strongly to zero, implying that it must not converge strongly at all.

• Given any bounded linear functional ϕ on $\ell^2(\mathbb{Z})$ and any $x \in \ell^2(\mathbb{Z})$, we can linearly decompose the action of ϕ on x as the action of the components ϕ_i of ϕ on the components x_i of x given by $\phi_i(x_j) = a_i x_j$ and find

$$\begin{aligned}
\phi(S_n x) =& \phi(0, \ldots, 0, x_1, x_2, \ldots) \\
=& \sum_{i=n+1}^{\infty} \phi_i(x_{i-n}) \\
=& \sum_{i=n+1}^{\infty} a_i x_{i-n} \\
\leq& \left(\sum_{j=1}^{\infty} x_j^2 \right)^{(1/2)} \left(\sum_{i=n+1}^{\infty} a_i^2 \right)^{(1/2)},
\end{aligned}$$

which goes to zero since $a_i \to 0$ as $n \to \infty$.

Thus, S_n converges weakly to 0. However, it is clear from the definition that $||S_n x|| = ||x||$ for all $n \in \mathbb{N}$. Therefore, we cannot have

$$S_n x \to 0 \quad \text{as } n \to \infty.$$

Hence, S_n converges weakly to 0, but not strongly.

□

2. A subset E of a vector space X is said to be convex if

$$\lambda x + (1 - \lambda) y \in E \quad \forall x, y \in E, 0 \leq \lambda \leq 1.$$

(a) Show that a strongly closed, convex subset of a Hilbert space is weakly closed.

Proof. If $x_n \rightharpoonup x$, where $\{x_n\}_n \subset E$, by *Mazur's theorem*, there is a sequence of $\{y_n\}_n \subset E$ of finite convex combination of $\{x_n\}_n$ such that $y_n \to x$. Note that the sequence $\{y_n\}_n$ is realy a subset of E because we were given that E is convex! So $x \in E$, because E is strongly closed. Therefore, we conclude that E is weakly closed.

□

(b) Show that every strongly closed, convex subset of a Hilbert space contains a point of minimum norm.

Proof. Suppose E is a closed convex set in a Hilbert space $\mathcal{H}$. Let

$$d = \inf_{x \in E} \|x\|.$$

• If $d = 0$, then we can find a sequence $\{x_n\}_n$ so that

$$\lim_n \|x_n\| = 0.$$

Then we find $\{x_n\}_n$ is convergent to 0. Therefore 0 is a limit point of E. Since E is closed we find $0 \in E$. If $\|x\| = 0$, then $x = 0$. Hence 0 is the unique minimum point when $d = 0$.

• Suppose $d > 0$. Then we can find a sequence $\{x_n\}_n$ so that $\lim_n \|x_n\| = d$.
By the parallelogram's law, we find

$$\|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - 4\left\|\frac{x_n + x_m}{2}\right\|^2. \tag{0.1}$$

Since E is convex, $x_n \in E$ for all $n \in \mathbb{N}$, then we find $(x_n + x_m)/2 \in E$. By the definition of d, we see that

$$d^2 \leq \left\|\frac{x_n + x_n}{2}\right\|^2. \tag{0.2}$$

(0.1) and (0.2) imply

$$\|x_n - x_m\|^2 \leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2.$$

Since $\lim_{n\to\infty} \|x_n\| = d$, (or $\lim_{n\to\infty} \|x_n\|^2 = d^2$), then for each $\epsilon > 0$, we can find $N_\epsilon > 0$ so that $n \geq N_\epsilon$,

$$0 \leq \|x_n\|^2 - d^2 < \frac{\epsilon^2}{4},$$

which implies that whenever $n, m \geq N_\epsilon$, we have

$$\|x_n - x_m\|^2 < \left(2d^2 + \frac{\epsilon^2}{2}\right) + \left(2d^2 + \frac{\epsilon^2}{2}\right) - 4d^2 = \epsilon^2.$$

We proved that $\{x_n\}_n$ is a Cauchy sequence.

• Since $\mathcal{H}$ is a Hilbert space, then we can find $x \in \mathcal{H}$ so that

$$\lim_{n\to\infty} x_n = x.$$

We find x is a limit point of E. Since E is closed, then $x \in E$.
We also have

$$\lim_{n\to\infty} \|x_n\| = \|x\| = d,$$

and this is given by the continuity of the norm. Hence x is indeed a minimum point.

• Suppose y is another point with $\|y\| = d$. Again, using the parallelogram's law, we find

$$\begin{aligned} 0 \le \|x-y\|^2 =& 2\|x\|^2 + 2\|y\|^2 - 4\left\|\frac{x+y}{2}\right\|^2 \\ =& 4d^2 - 4\left\|\frac{x+y}{2}\right\|^2 \\ \le& 4d^2 - 4d^2 = 0. \end{aligned}$$

We find $\|x-y\| = 0$, which implies that $x = y$. We conclude that the minimum point is unique. $\square$

And alternative proof is to observe that a strongly closed, convex set is weakly closed, and the norm is weakly lower semi-continuous and coercive, so it attains its minimum on any weakly closed set.