201B, Winter '11, Professor John Hunter
Homework 9 Solutions

1. If $A : \mathcal{H} \to \mathcal{H}$ is a bounded, self-adjoint linear operator, show that

$$\|A^n\| = \|A\|^n$$

for every $n \in \mathbb{N}$. (You can use the results proved in class.)

Proof.
• Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint linear operator. Then we can prove that A^k is also a bounded self-adjoint operator for any $k \in \mathbb{N}$. Why? The answer is because of the following argument.
Assume that $\|A\| = M$, where M is some constant greater than zero. Then by induction we can conlude the following:

$$\|A^n\| = \|A^{n-1}A\| \leq \|A^{n-1}\|\|A\| \leq \|A\|^n = M^n$$

and

$$(A^n)^* = (A^{n-1}A)^* = A^*(A^{n-1})^* = A(A^{n-1}) = A^n.$$

Hence, we got that indeed A^n is also a bounded, self-adjoint linear operator. The linearity of A^k follows right away from the linearity of A.

• The spectral radius of a bounded self-adjoint operator is given by its norm; more precisely, remember that

$$\|A^k\| = r(A^k) = \lim_{n\to\infty} \|A^{kn}\|^{1/n}.$$

Relabelling the indices so that $u = nk$, we have that

$$\begin{aligned}\|A^k\| &= \lim_{u\to\infty} \|A^u\|^{k/u} \\ &= \left(\lim_{u\to\infty} \|A^u\|^{1/u}\right)^k \\ &= r(A)^k \\ &= \|A\|^k.\end{aligned}$$

Hence, we are done proving our problem. □

2. Define $m : (0,1) \to (0,1)$ be

$$m(x) = \begin{cases} 0 & \text{if } 0 < x < 1/4 \\ 2(x - 1/4) & \text{if } 1/4 \leq x \leq 3/4 \\ 1 & \text{if } 3/4 < x < 1 \end{cases}$$

and let $M : L^2(0,1) \to L^2(0,1)$ be the multiplication operator $Mf = mf$. Determine the spectrum of M and classify it into point, continuous and residual spectrum. Describe the eigenspace of any eigenvalues in the point spectrum.

Proof. due to Amanda.
• Fisrt let's establish some properties about the operator M. Clearly, M is a self-adjoint operator since we have the following:

$$\langle f,\, Mg\rangle = \int_0^1 \overline{f(x)} m(x) g(x) dx = \int_0^1 \overline{m(x) f(x)} g(x) dx = \langle Mf,\, g\rangle.$$

Note that here the fact that $m(x)$ is a real valued function, i.e., $\overline{m(x)} = m(x)$, helped us to conclude that M is self-adjoint.

• M is self-adjoint, implies that $\sigma_r(M) = \emptyset$ and $\sigma(M) \subseteq [-\|M\|, \|M\|]$. This results that I have just stated can be found in *Prop.9.8* and *Lemma 9.13* form your textbook.

• Looking at the definition of $m(x)$, we can see that the maximum value that m can attain is 1 and we can also observe that $0 \leq m^2(x) \leq 1$ for all $x \in [0,1]$.
Hence, applying the definition of the norm of M we get:

$$\begin{aligned} \|M\| &= \sup_{\|f\|=1} \|Mf(x)\| \\ &= \sup_{\|f\|=1} \left(\int_0^1 m^2(x) |f(x)|^2 dx \right)^{1/2} \\ &\leq \sup_{\|f\|=1} \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} \\ &= 1 \end{aligned}$$

• If we take any $f(x) \in L^2(0,1)$ with support on $(3/4,1)$ such that $\|f(x)\| = 1$, we see that $\|Mf(x)\| = 1$, so $\|M\| = 1$. Hence, $\sigma(M) \subseteq [-1,1]$. Notice that for any nonzero function $f(x)$ with support on $(0,1/4)$ that $Mf(x) = 0$, so these are functions with eigenvalue 0.

Furthermore, we know such functions exist. Similarly, any nonzero function $f(x)$ with support on $(3/4,1)$ satisfies $Mf(x) = f(x)$ so these are functions with eigenvalue 1. So $\{0,\, 1\} \subseteq \sigma_p(M)$.

• Consider any $\lambda \in [-1, 0)$. Then since $m(x) \geq 0$ for all x, it follows that $m(x) - \lambda \geq -\lambda > 0$ for all x. Since $(M - \lambda I)f(x) = (m(x) - \lambda)f(x)$, if $(M - \lambda I)f(x) = 0$ then $f(x) = 0$ a.e., so it follows that $M - \lambda I$ is one-to-one.

Also, if $g(x) \in L^2(0, 1)$ then $\frac{1}{m(x)-\lambda} g(x)$ is well-defined and in $L^2(0, 1)$, so

$$(M - \lambda I)\frac{1}{m(x) - \lambda} g(x) = g(x).$$

So $M - \lambda I$ is also onto.

Hence, for $\lambda \in [-1, 0)$ we see that $\lambda \in \rho(M)$, so $\sigma(M) \subseteq [0, 1]$. If $\lambda \in (0, 1)$, then we still have $M - \lambda I$ is one-to-one since $m(x) - \lambda$ is nonzero for all but one value of x so $(M - \lambda I)f(x) = (m(x) - \lambda)f(x) = 0$ implies $f(x) = 0$.

However, $M - \lambda I$ is not onto. If it were onto, then $1 \in \operatorname{ran}(M - \lambda I)$. Then there would be an $f(x) \in L^2(0, 1)$ such that

$$(M - \lambda I)f(x) = (m(x) - \lambda)f(x) = 1 \qquad \text{so} \qquad f(x) = \frac{1}{m(x) - \lambda}$$

However, there is some $x_0 \in (1/4, 3/4)$ such that $m(x_0) = 2(x_0 - 1/4) = \lambda$.

It follows that

$$\int_{1/4}^{3/4} \frac{1}{(2(x - 1/4) - \lambda)^2} dx \to \infty.$$

Since

$$\|f(x)\|_2^2 = \int_0^{1/4} \frac{1}{\lambda^2} dx + \int_{1/4}^{3/4} \frac{1}{(2(x - 1/4) - \lambda)^2} dx + \int_{3/4}^{1} \frac{1}{(1 - \lambda)^2} dx$$

we see that $f(x) \notin L^2(0, 1)$. Hence, $M - \lambda I$ is not onto.

Therefore, $\lambda \in \sigma(M)$ for $\lambda \in (0, 1)$. Particularly, since $M - \lambda I$ is one-to-one, we know that $\lambda \in \sigma_c(M)$ or $\lambda \in \sigma_r(M)$. From class we know that $\sigma_r(M) = \emptyset$ since M is self-adjoint. Hence, it must be that $\lambda \in \sigma_c(M)$,

Therefore, the spectrum $\sigma(M) = [0, 1]$ where $\sigma_p(M) = \{0,\, 1\}$ and $\sigma_c(M) = (0, 1)$.

□

3 Suppose that $\{\lambda_n\}$ is a sequence of complex numbers such that $\lambda_n \to 0$ as $n \to \infty$ and define the operator $K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$K(x_1,\, x_2, \ldots,\, x_n,\, \ldots) = (\lambda_1 x_1,\, \lambda_2 x_2, \ldots,\, \lambda_n x_n,\, \ldots)$$

(a) Prove that K is a compact operator. (Recall that a set is precompact iff it is totally bounded)

Proof. Let $\{\lambda_n \to 0\}$ and let $K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined by

$$K(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots).$$

We want to show that K is a compact operator. We know that finite-rank operators are compact operators (you have proved this as a homework problem in 201A) and we also know that the uniform limit of compact operators is a compact operator.
Let

$$K_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

be given by

$$K_n(x_1, x_2, \ldots) = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, 0, 0, \ldots).$$

Then K_n is finite-rank operator and hence, a compact operator.
Note that

$$\begin{aligned}
||K - K_n|| &= \sup_{||x||=1} ||(K - K_n)x|| \\
&= \sup_{||x||=1} ||(0, \ldots, 0, \lambda_{n+1} x_{n+1}, \lambda_{n+2} x_{n+2}, \ldots)|| \\
&= \sup_{\|x\|=1} \sum_{i=n+1}^{\infty} |\lambda_i x_i|^2 \\
&\leq \sup_{\|x\|=1} \sum_{i=n+1}^{\infty} |\lambda_i|^2 |x_i|^2 \\
&\leq \sup_{\|x\|=1} \sup_{j=n+1} |\lambda_j|^2 \sum_{i=n+1}^{\infty} |x_i|^2 \\
&\leq \sup_{\|x\|=1} \sup_{j=n+1} |\lambda_j|^2 \sum_{i=1}^{\infty} |x_i|^2 \\
&\leq \sup_{\|x\|=1} \sup_{j=n+1} |\lambda_j|^2 \|x\|^2 \\
&\leq \sup_{j=n+1} |\lambda_j|^2,
\end{aligned}$$

which can be made arbitrarily small for sufficiently large n, i.e, $\to 0$ as $n \to \infty$.

Therefore we can conclude that $K_n \to K$ uniformly and K is compact.

□

(b) Let $P_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the orthogonal projection onto the nth component,

$$P_n(x_1, x_2, \ldots, x_n, \ldots) = (0, 0, \ldots, 0\,, x_n, 0, \ldots)$$

In what sense (uniformly, strongly, weakly) does the sum $\sum_{n\in\mathbb{N}} \lambda_n P_n$ converge to K? Does your answer change in $\lambda \not\to 0$ as $n \to \infty$

Proof. • Since $P_n(x_1, x_2, \ldots) = (0, \ldots, 0, x_n, 0, \ldots)$, using the same K_n as the one we defined above, we have

$$K_n = \sum_{k=1}^{n} \lambda_n P_n.$$

We have probed above that $K_n \to K$ uniformly.
• Now we discuss the case when λ_n does not converge to 0, but it is still bounded.

♡ In this case we can see that K_n doesn't converge uniformly to K, and the reason is the following:

$$\begin{aligned}\lim_{n\to\infty} ||K - K_n|| &= \lim_{n\to\infty} \sup_{\|x\|=1} \sum_{i=n+1}^{\infty} |\lambda_i x_i|^2 \\ &\leq \lim_{n\to\infty} \sup_{\|x\|=1} \sum_{i=n+1}^{\infty} |\lambda_i|^2 |x_i|^2;\end{aligned}$$

and since we assumed that $\lambda_n \nrightarrow 0$ then there exists an $\epsilon > 0$ such that for every $M \in \mathbb{N}$ we can find $m > M$ such that $|\lambda_m|^2 > \epsilon$. Picking $x = e_m$, we get that

$$\lim_{n\to\infty} ||K - K_n|| > \epsilon.$$

This proves that indeed K_n doesn't converge uniformly to K.

♡ In the same context, meaning the case when λ_n does not converge to 0, but it is still bounded, we want to see if K_n converges strongly to K. For this, let's fix $x \in \ell^2(\mathbb{N})$. Assume also that $|\lambda_i| \leq G$ for every $i \in \mathbb{N}$, where G is a positive constant.

Therefore

$$\lim_{n\to\infty} ||(K - K_n)x|| = \lim_{n\to\infty} \sum_{i=n+1}^{\infty} |\lambda_i x_i|^2$$

$$\leq \lim_{n\to\infty} G^2 \sum_{i=n+1}^{\infty} |x_i|^2.$$

But, this last term converges to 0 as $n \to \infty$, since $x \in \ell^2(\mathbb{N})$, by definition implies

$$\sum_{i=n+1}^{\infty} |x_i|^2 \leq \sum_{i=1}^{\infty} |x_i|^2 \to 0 \text{ as } n \to \infty.$$

Hence, $K_n \to K$ strongly. This clearly implies that $K_n \to K$ weakly.

• If we assume that the sequence $\{\lambda_n\}_n$ is not bounded either, then from the work we did above, we can see that $K_n \nrightarrow K$ weakly. Implicitly this tells you that K_n doesn't converge strongly or uniformly to K.

□

4. Determine the spectra of the left and right shift operators on $\ell^2(\mathbb{N})$

$$S(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots),$$
$$T(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),$$

and classify them into point, continuous, or residual spectrum.

Proof. due to Amanda.
We have previously shown that $\|S\| = 1 = \|T\|$. So by a theorem if $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$, then $\lambda \in \rho(S)$ and $\lambda \in \rho(T)$. Now I will prove the following four claims:

(a) Claim: $S - \lambda I$ is one-to-one for all $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$.

Proof. I will treat this in two cases. If $\lambda = 0$ then $Sx = 0$ implies $x_i = 0$ for all i so $x = 0$. Hence, $S - \lambda I$ is one-to-one. Suppose that $0 < |\lambda| \leq 1$. Then $(S - \lambda I)x = 0$, implies

$$(0, x_1, x_2, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots)$$

Since $\lambda \neq 0$, $0 = \lambda x_1$ implies $x_1 = 0$. Since $\lambda x_n = x_{n-1}$, a simple induction shows that $x_n = 0$ for all n. Hence, $x = 0$, so $S - \lambda I$ is one-to-one for all λ such that $|\lambda| \leq 1$.

□

(b) Claim: $S - \lambda I$ is not onto for $|\lambda| \leq 1$.

Proof. Note that if $\lambda = 0$ that e_1 is clearly not in $\operatorname{ran}(S - \lambda I) = \operatorname{ran}(S)$. Suppose that $\lambda \neq 0$. Then $(S - \lambda I)x = e_1$ implies that $-\lambda x_1 = 1$ and $x_{n-1} - \lambda x_n = 0$ for $n \geq 2$. An induction argument shows that $x_1 = -1/\lambda$, and $x_n = -1/\lambda^n$. However,

$$\|x\| = \sum_{n=1}^{\infty} \left(\frac{1}{|\lambda^2|}\right)^n$$

Note the above sum is a geometric series with $r = \frac{1}{|\lambda|^2}$. Since $0 < |\lambda|^2 \leq 1$, it follows that $\frac{1}{|\lambda|^2} \geq 1$. Hence the above sum diverges, so $x \notin \ell^2(\mathbb{N})$. Therefore, $e_1 \notin \operatorname{ran}(S - \lambda I)$ so this is not onto.

□

(c) Claim: $T - \lambda I$ is one-to-one for $|\lambda| = 1$.

Proof. Suppose that it was not one-to-one. Then there would be some nonzero $x \in \ell^2(\mathbb{N})$ such that $(T - \lambda I)x = 0$ or

$$(x_2, x_3, x_4, \ldots) = (\lambda x_1, \lambda x_2, \lambda x_3, \ldots)$$

Hence, $x_2 = \lambda x_1$ and a simple induction shows that $x_n = \lambda^{n-1} x_1$. Therefore, $x \neq 0$ implies $x_1 \neq 0$, and since $|\lambda| = 1$ we see that

$$\|x\| = \sum_{n=0}^{\infty} |x_1|^2 |\lambda|^{2n} = \sum_{n=0}^{\infty} |x_1|^2 \to \infty$$

a contradiction so it must be that $x = 0$, and it follows that $T - \lambda I$ is one-to-one. □

(d) Claim: $T - \lambda I$ is not one-to-one for $|\lambda| < 1$.

Proof. By a similar argument as above, we see that any nonzero x that satisfies $(T - \lambda I)x = 0$ is of the form

$$x = (x_1, \lambda x_1, \lambda^2 x_1, \lambda^3 x_1, \ldots)$$

Choose $x_1 = 1$. Then

$$\|x\| = \sum_{n=0}^{\infty} (|\lambda|^2)^n$$

Since the above is a geometric series with $r = |\lambda|^2 < 1$, we see that the sum converges, so $x \in \ell^2(\mathbb{N})$ is nonzero and satisfies that $(T - \lambda I)x = 0$. Hence, $T - \lambda I$ is not one-to-one. □

Note that claims (a) and (b) show that $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Recall that $\operatorname{ran}(S - \lambda I)$ is dense iff $\ker((S - \lambda I)^*) = \ker(T - \overline{\lambda} I) = \{0\}$. Since $|\overline{\lambda}| = |\lambda|$, claims (c) and (d) show that $\operatorname{ran}(S - \lambda I)$ is dense iff $|\lambda| = 1$. Therefore, $\sigma_c(S) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma_r(S) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Also, from the above work we get $\sigma_p(S) = \emptyset$.

Note that claim (d) shows us that $\sigma_p(T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Since $\sigma(T)$ is a closed set, we know that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ (as from above we know that $|\lambda| > 1$ implies $\lambda \in \rho(T)$). For $|\lambda| = 1$ we know that $\ker((T - \lambda I)^*) = \ker(S - \overline{\lambda} I) = \{0\}$ by claim (a). Hence, we know that $\operatorname{ran}(T - \lambda I)$ is dense. Therefore, it must be that $\sigma_c(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $\sigma_r(T) = \emptyset$.

□