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ON THE APPLICATION OF BOREL'S METHOD TO THE SUMMATION OF FOURIER'S SERIES[1]

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There are two ways of studying the relationship between various methods for the summation of divergent series. One consists in the attempt to determine directly whether or not one is more general than the other, and if this is not the case to determine under what conditions both methods apply and give the same sum to the series that is used. Another method involves the determination of the relative scope of the various processes in summing certain general types of series that are of fundamental importance in analysis. The former method is more exhaustive from the theoretical point of view; the latter is, perhaps, of greater practical interest.

The two most important types of series in analysis at the present time are power series and Fourier's series. It is well known that Cesàro's method will not sum a power series outside of its circle of convergence, whereas Borel's method applies everywhere within the polygon of summability. However, in the case where the circle of convergence is a natural boundary, Cesàro's method may be applicable at points on the circle of convergence where Borel's method fails. This phase of the relationship between the two methods may well be described by an illuminating remark made by G. H. Hardy[2] in another connection, namely, that "Borel's method, although more powerful than Cesàro's, is never more delicate, and often less so."

Cesàro's method has been found to be admirably adapted to the study of Fourier's series. It will give the proper sum for the Fourier's series of any continuous function at all points, and will sum the Fourier's series of any function having a Lebesgue integral to the value of the function, except perhaps at a set of points of measure zero. Since there is considerable similarity in the behavior of Fourier's series and the behavior of power series on the circle of convergence, it is natural to expect that Borel's method will not be as effective as Cesàro's in dealing with the former type of series. That this is the case is proved by the results of the present paper.

The application of Borel's method to the summing of Fourier's series leads to the study of the behavior of the following integral[3]

$$\frac{1}{\pi}\int_0^{\frac{\pi}{2}} [f(x + 2t) + f(x-2t)]e^{-Xt^2}\,\frac{\sin Xt}{t}\,dt$$

as X becomes infinite. This integral is of the type termed by Lebesgue singular integrals. Its kernel

$$\varphi(X, t) = \frac{2}{\pi}\,e^{-Xt^2}\,\frac{\sin Xt}{t}$$

is such that

$$L_B(X) = \int_0^{\frac{\pi}{2}} |\,\varphi(X, t)\,|\,dt \qquad (1)$$

does not remain bounded as X becomes infinite. It follows therefore from a general theorem due to Lebesgue[4] that Borel's method will not sum the Fourier's series of every function having a Lebesgue integral or even of every continuous function.

Having found that Borel's method is less effective than Cesàro's in summing Fourier's series, we next wish to determine if it is more effective than ordinary convergence. The kernel of the singular integral that arises in the study of convergence, the well known Dirichlet's integral, is

$$\psi(n, t) = \frac{2}{\pi}\cdot\frac{\sin\,(2n + 1)t}{\sin t}.$$

The values of

$$L(n) = \int_0^{\frac{\pi}{2}} |\,\psi(n, t)\,|\,dt \qquad (2)$$

for successive values of n are termed Lebesgue constants, this designation having been introduced by Fejér and adopted by later writers. The fact that they become infinite with n is tied up with the fact that the Fourier's series of a continuous function may diverge, and their order of infinity may be regarded as one form of measure of the degree of divergence that is possible in the case of the Fourier's series of a continuous function. The values for odd integral values of X of the function $L_B(X)$, defined by (1), may be regarded as the analogues of the Lebesgue constants for the application of Borel's method of summation. In spite of the factor e^{-Xt^2} in the integrand of (1), it may be shown that $L_B(n)$ is of the same order of infinity as $L(n)$, this order being that of $\log n$.

In view of this fact, one might not be surprised to find that Borel's method was no more effective in summing the Fourier's series of a continuous function than ordinary convergence. However, this is not the case, as can be shown by exhibiting an example of a continuous function whose Fourier's series is divergent but can nevertheless be summed by Borel's method. This example is obtained by slightly modifying an example given by Fejér[5] of a function whose Fourier's series diverges. Fejér's function is

$$\theta(x) = \sum_{n=1}^{\infty} \frac{\sin 2^{n^3} x}{n^2} \qquad (0 \leqq x \leqq \pi).$$

The cosine development of this function diverges for $x = 0$, and cannot be summed at this point by Borel's method. If, however, we set

$$\alpha(x, n) = 2^{\frac{1}{4}n^3} x \sin 2^{\frac{3}{4}n^3} \qquad (0 \leqq x \leqq 2^{-\frac{1}{4}n^3}),$$

$$\alpha(x, n) = \sin 2^{n^3} x \qquad (2^{-\frac{1}{4}n^3} < x \leqq \pi),$$

and form

$$F(x) = \sum_{n=1}^{\infty} \frac{\alpha(x, n)}{n^2} \qquad (0 \leqq x \leqq \pi),$$

we obtain a function whose cosine development diverges for $x = 0$ but can nevertheless be summed at that point by Borel's method.

The underlying reason for these facts may be briefly stated as follows. The essential cause for the divergence as well as the failure of Borel summability in the case of the Fourier's development of Fejér's function lies in the degree of rapidity with which the oscillations in the neighborhood of the origin of the individual terms of the series defining this function increase in frequency as we pass from one term to the next. The effect of the modifications we have made in the series defining Fejér's function is to remove such of the oscillations of each term as lie in a certain neighborhood of the origin, this neighborhood becoming steadily smaller as we pass to later terms. Thus we find in the resulting function oscillations of higher and higher frequency sufficiently near to the origin to cause divergence of the Fourier's development at that point. On the other hand we have removed enough of the oscillations in the immediate neighborhood of the origin to attain Borel summability of the development at that point, since the factor e^{-Xt^2} in the corresponding singular integral neutralizes the effect of the others.

It is apparent that the foregoing discussion, in addition to shedding further light on the relation between Borel and Cesàro summability, also furnishes one scheme for distinguishing between various types of continuous functions that have divergent Fourier's developments. It may be added that the scope of Euler's method[6] in the summation of Fourier's series is the same as that of Borel's method, the resulting singular integrals being essentially equivalent. Hence the Fourier's development of the function we have defined above will also be summable by Euler's method at the point $x = 0$.

[1] Presented to the American Mathematical Society, December 29, 1924.

[2] Hardy, G. H., *Proc. London Math. Soc.*, Ser. 2, **11,** 1911 (1–16), p. 10.

[3] The integral first obtained is more complex but is essentially equivalent to the one given here.

[4] Lebesgue, H., *Toulouse Annales*, Ser. 3, **1,** 1910 (25–128), p. 70.

[5] Fejér, L., *Crelle's Journal*, **137,** 1910 (1–5).

[6] Cf. Knopp, K., *Math. Zeitschrift*, **15,** 1922 (226–253).

INTERSECTIONS OF COMPLEXES ON MANIFOLDS

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1. Let M_n, C_h, C_k be an orientable manifold and two complexes on it, the indices denoting the dimensions.[1] We propose to discuss what is meant in the most general case by the intersection of the complexes, and in particular by their Kronecker index when $h + k = n$.

We assume M_n covered with a C_n to be used as a basis for the definition of straightness and distances. If then C_h and C_k are made up exclusively with straight elements intersecting each other in the most general way possible, there is no essential difficulty. For straight cells in general relative position a procedure outlined elsewhere is applicable.[2] If the h and k cells of the complexes so behave, then the extension to them is also immediate, the sensed intersection being denoted by $C_h . C_k$, the Kronecker index when $k = n-h$, by $(C_h . C_k)$. Let Γ_{h-1}, Γ_{k-1} be the boundary cycles. We may then prove the basic Poincaré congruence

$$C_h . C_k \equiv (-1)^{n-k}\, \Gamma_{h-1} . C_k + C_h . \Gamma_{k-1}.$$

2. When the complexes are arbitrary our discussion will lead us only to a definition of a clear cut intersection if their boundaries do not intersect one another, and then $C_h . C_k$ is a cycle whose exact determination is obtained thus: We approximate C_h as closely as we please by a polyhedral complex C'_h such that there exist C_{h+1} and $\overset{\circ}{C}_h$, respectively, very near C_h and its boundary, with a congruence

$$C_{h+1} \equiv C_h + \overset{\circ}{C}_h - C'_h. \tag{1}$$

Moreover, $\overset{\circ}{C}_h$ contains the boundaries of both C_h and C'_h. This may