Final Solutions: Math 201B
Winter, 2011

1. (a) Show that there is a unique solution $G \in \mathcal{D}'(\mathbb{T})$ of the ODE

$$-G'' + G = \delta,$$

where $\delta \in \mathcal{D}'(\mathbb{T})$ is the periodic delta-function supported at 0, and compute the Fourier series of G.

(b) Define the Sobolev space $H^s(\mathbb{T})$ for real numbers $s \geq 0$. For what $s \geq 0$ is it true that $G \in H^s(\mathbb{T})$?

Solution.

- (a) Any distribution $G \in \mathcal{D}'(\mathbb{T})$ may be expanded in a Fourier series

$$G(x) = \sum_{n\in\mathbb{Z}} \hat{G}(n)e^{inx}, \qquad \hat{G}(n) = \frac{1}{2\pi}\left\langle G, e^{-inx}\right\rangle$$

where the Fourier series converges in the sense of distributions. Moreover, the coefficients $\hat{G}(n)$ are the Fourier coefficients of a distribution if and only if they have slow growth as $n \to \infty$.

- Since the differentiation operation is continuous on $\mathcal{D}'(\mathbb{T})$ and

$$\left(e^{inx}\right)' = ine^{inx}$$

we have

$$-G'' + G = \sum_{n\in\mathbb{Z}} \left(n^2+1\right)\hat{G}(n)e^{inx}.$$

- The delta-function has the Fourier series

$$\delta(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{inx}.$$

- Two distributions are equal if and only if their Fourier coefficients are equal, so G is a solution of the ODE if

$$\left(n^2+1\right)\hat{G}(n) = \frac{1}{2\pi}$$

or

$$G(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} \frac{1}{n^2+1}e^{inx}. \tag{1}$$

Conversely, this Fourier series defines a distributional solution since the Fourier coefficients $\hat{G}(n)$ have slow growth (in fact they decay). Thus the ODE has the unique distributional solution (1).

- (b) The space $H^s(\mathbb{T})$ consist of functions $f \in L^2(\mathbb{T})$ with Fourier series

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$$

such that

$$\sum_{n\in\mathbb{Z}} \left(1+n^2\right)^s \left|\hat{f}(n)\right|^2 < \infty.$$

- The function G in (1) belongs to $H^s(\mathbb{T})$ if

$$\sum_{n\in\mathbb{Z}} \frac{1}{(1+n^2)^{2-s}} < \infty$$

which is the case if $2(2-s) > 1$ or $s < 3/2$.

Remark. The function G is the Green's function of the ODE

$$Lu = f, \qquad L = -\frac{d^2}{dx^2} + 1$$

with periodic boundary conditions. The solution is given by $u = G * f$, so that convolution with G gives L^{-1}. Explicitly,

$$u(x) = \int_{\mathbb{T}} G(x-y) f(y)\, dy.$$

The "physical" interpretation of this result (which was the origin of the delta-function) is that G is the response of the system to a point source δ, and the response for a general source

$$f(x) = \int_{\mathbb{T}} \delta(x-y) f(y)\, dy$$

is obtained by superposing the corresponding point source responses by linearity.

2. (a) Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded self-adjoint linear operator on a Hilbert space $\mathcal{H}$ which leaves a linear subspace $\mathcal{M} \subset \mathcal{H}$ invariant, meaning that $A : \mathcal{M} \to \mathcal{M}$. Prove that A leaves the orthogonal complement $\mathcal{M}^\perp$ invariant.

(b) Give an example of a non-selfadjoint operator $A : \mathbb{C}^2 \to \mathbb{C}^2$ for which this result is not true.

Solution.

- (a) Suppose that $y \in \mathcal{M}^\perp$. Then

  $$\langle y, z\rangle = 0 \qquad \text{for every } z \in \mathcal{M}.$$

  In particular, since $Ax \in \mathcal{M}$ for every $x \in \mathcal{M}$, it follows that

  $$\langle y, Ax\rangle = 0 \qquad \text{for every } x \in \mathcal{M}.$$

  Since A is self-adjoint we have $\langle y, Ax\rangle = \langle Ay, x\rangle$, so that

  $$\langle Ay, x\rangle = 0 \qquad \text{for every } x \in \mathcal{M},$$

  which means that $Ay \in \mathcal{M}^\perp$. Therefore A leaves $\mathcal{M}^\perp$ invariant.

- (b) For $\lambda \in \mathbb{C}$, let A be the linear transformation on $\mathbb{C}^2$, with the standard inner product, whose matrix is

  $$[A] = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

  Then

  $$\mathcal{M} = \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} : a \in \mathbb{C} \right\}.$$

  is an invariant subspace of A, but

  $$\mathcal{M}^\perp = \left\{ \begin{pmatrix} 0 \\ b \end{pmatrix} : b \in \mathbb{C} \right\}$$

  is not invariant.

Remark. Part (a) is the basic result that allows us to reduce the action of a self-adjoint operator to smaller and smaller subspaces in developing their spectral theory.

3. Suppose that K is a compact, but not necessarily self-adjoint, linear operator on a Hilbert space $\mathcal{H}$.

(a) Prove that $I + K$ has closed range (where I is the identity operator).

(b) State a necessary and sufficient condition on $y \in \mathcal{H}$, in terms of K^*, for the solvability of the equation

$$(I + K)x = y. \tag{2}$$

Solution.

- (a) Write $A = I + K$ and let

  $$\tilde{A} : \mathcal{K} \to \mathcal{H}, \qquad \mathcal{K} = (\ker A)^\perp$$

  be the restriction of A to the closed subspace $(\ker A)^\perp$. Then $\tilde{A}$ is one-to-one and $\operatorname{ran} \tilde{A} = \operatorname{ran} A$; because any $x \in \mathcal{H}$ may be written as $x = y + z$ where $y \in \ker A$ and $z \in (\ker A)^\perp$, so $Ax = \tilde{A}z$.

- If A is one-to-one, then $\tilde{A} = A$, but this need not be true in general, since K may have -1 as an eigenvalue.

- The range of $\tilde{A}$, and therefore the range of A, is closed if there exists a constant $c > 0$ such that

  $$c\|x\| \leq \left\|\tilde{A}x\right\| \qquad \text{for all } x \in \mathcal{K}. \tag{3}$$

  (Proof: If $y_n = \tilde{A}x_n \in \operatorname{ran} \tilde{A}$ and $y_n \to y$ in $\mathcal{H}$, then $\{y_n\}$ is Cauchy, so $\{x_n\}$ is Cauchy from (3); hence $x_n \to x \in \mathcal{K}$ and $y = \tilde{A}x \in \operatorname{ran} \tilde{A}$ since $\tilde{A}$ is bounded.)

- Suppose, for contradiction, that (3) is false. Then there exists a sequence $\{x_n\}$ in $\mathcal{K}$ such that $\|x_n\| = 1$ and

  $$\left\|\tilde{A}x_n\right\| \to 0.$$

- Since K is compact and $\{x_n\}$ is bounded there is a subsequence, which we still denote by $\{x_n\}$, such that Kx_n converges, to $z \in \mathcal{H}$ say. It follows that

  $$x_n = \tilde{A}x_n - Kx_n \to -z$$

  also converges. Moreover, $z \in \mathcal{K}$ since $\{x_n\}$ is in $\mathcal{K}$ and $\mathcal{K}$ is a closed linear subspace.

- We conclude that $\|z\| = 1$, since $\|x_n\| = 1$ for every n, and

$$\tilde{A}z = -\lim_{n\to\infty} \tilde{A}x_n = 0,$$

which contradicts the fact that $\tilde{A}$ is one-to-one. This contradiction proves that (3) holds, so A has closed range.

- (b) For any bounded linear operator A, we have

$$\mathcal{H} = \overline{\operatorname{ran} A} \oplus \ker A^*.$$

Since $\operatorname{ran}(I + K)$ is closed, (2) is solvable for x if and only if

$$y \perp \ker(I + K^*),$$

meaning that

$$\langle z, y\rangle = 0 \qquad \text{for all } z \in \mathcal{H} \text{ such that } z + K^*z = 0.$$

Remark. An analogous result to (a) is true for compact operators on a Banach space X. The proof is similar, with the quotient space $X/\ker A$ replacing the orthogonal complement $(\ker A)^\perp$.

4. Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space.

(a) State the spectral theorem for compact, self-adjoint linear operators on $\mathcal{H}$. (You can state the theorem in any form you want provided you state it precisely and completely.)

(b) Prove that two compact, self-adjoint linear operators A, B on $\mathcal{H}$ are unitarily equivalent (meaning that there is a unitary operator U on $\mathcal{H}$ such that $A = U^*BU$) if and only if

$$\dim \ker(\lambda I - A) = \dim \ker(\lambda I - B)$$

for every $\lambda \in \mathbb{C}$.

(c) Does the result in (b) remain true if A, B are not both assumed to be self-adjoint?

Solution.

- (a) See text.
- (b) The assumption implies that A, B have the same eigenvalues $\{\lambda_n \in \mathbb{R} : n \in \mathbb{N}\}$ with the same multiplicities (possibly countably infinite in the case of $\lambda = 0$). Let $\{\lambda_n : n \in \mathbb{N}\}$ denote the common eigenvalues, repeated according to their multiplicity. By the spectral theorem for compact self-adjoint operators, both A and B have a complete orthonormal set of corresponding eigenfunctions, which we denote by $\{\phi_n \in \mathcal{H} : n \in \mathbb{N}\}$ and $\{\psi_n \in \mathcal{H} : n \in \mathbb{N}\}$, respectively.
- Define $U : \mathcal{H} \to \mathcal{H}$ to be the linear map such that $U\phi_n = \psi_n$. Since U maps an orthonormal basis of $\mathcal{H}$ to an orthonormal basis, it is a unitary map with $U^* = U^{-1}$.
- We have
$$BU\phi_n = B\psi_n = \lambda_n\psi_n = U(\lambda_n\phi_n) = UA\phi_n$$
It follows that $BU = UA$ and $A = U^*BU$.
- (c) The result is not true if A, B are not necessarily self-adjoint. For example, consider the maps $A, B : \mathbb{C}^3 \to \mathbb{C}^3$ with matrices
$$[A] = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \qquad [B] = \begin{pmatrix} \mu & 1 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

where λ, μ are distinct complex numbers. (The maps may be extended by 0 on $(\mathbb{C}^3)^\perp$ to maps on an infinite-dimensional space.) Then λ, μ each have geometric multiplicity one for both A and B, and all other complex numbers have multiplicity zero, but A and B are not unitarily equivalent; in fact, they are not even similar since they have different Jordan normal forms.

Remark. This result shows that compact self-adjoint operators are determined (up to unitary equivalence) by their spectrum and their spectral multiplicity; non-self-adjoint operators with the same spectrum and multiplicity may have different structures.

5. Let $E \subset H^1(\mathbb{T})$ be a bounded subset of the Sobolev space $H^1(\mathbb{T})$, meaning that there exists a constant M such that $\|f\|_{H^1} \leq M$ for all $f \in E$. Prove that E is a precompact subset of $L^2(\mathbb{T})$.

Solution.

- If $f \in H^1(\mathbb{T})$, then f is absolutely continuous and

$$f(x) - f(y) = \int_y^x f'(t)\, dt. \tag{4}$$

  The Cauchy-Schwartz inequality implies that every $f \in E$ satisfies

$$\begin{aligned} |f(x) - f(y)| &= \left| \int_y^x 1 \cdot f'(t)\, dt \right| \\ &\leq \left| \int_y^x 1^2\, dt \right|^{1/2} \left| \int_y^x |f'(t)|^2\, dt \right|^{1/2} \\ &\leq M\, |x - y|^{1/2} \end{aligned}$$

  It follows that E is equicontinuous.

- Integrating (4) with respect to y over $\mathbb{T}$, we get for $0 \leq x \leq 2\pi$ that

$$2\pi f(x) - \int_0^{2\pi} f(y)\, dy = \int_0^{2\pi} \left(\int_y^x f'(t)\, dt \right) dy.$$

  An integration by parts on the right-hand side (which is valid when f is absolutely continuous) gives

$$2\pi f(x) - \int_0^{2\pi} f(y)\, dy = 2\pi \int_{2\pi}^x f'(t)\, dt + \int_0^{2\pi} y f'(y)\, dy.$$

- Using the Cauchy-Schwartz inequality again, we get (without attempting to make the inequality sharp)

$$\begin{aligned} |f(x)| &\leq \frac{1}{2\pi} \left| \int_0^{2\pi} f(y)\, dy \right| + \left| \int_{2\pi}^x f'(t)\, dt \right| + \frac{1}{2\pi} \left| \int_0^{2\pi} y f'(y)\, dy \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \left(\int_0^{2\pi} |f(y)|^2\, dy \right)^{1/2} + \sqrt{2\pi} \left(\int_0^{2\pi} |f'(t)|^2\, dt \right)^{1/2} \\ &\qquad + \sqrt{2\pi} \left(\int_0^{2\pi} |f'(y)|^2\, dy \right)^{1/2} \\ &\leq 7M \end{aligned}$$

  It follows that E is bounded.

- Since E is equicontinuous and bounded, the Arzelà-Ascoli theorem implies that every sequence in E contains a uniformly convergent subsequence, and hence a subsequence that converges in $L^2(\mathbb{T})$. Therefore E is precompact in $L^2(\mathbb{T})$.

Remark. This result is a basic example of a compact Sobolev embedding: The inclusion map $J : H^1(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$ is compact, and a uniform bound on the derivatives of a bounded sequence of functions implies the strong convergence of a subsequence.