Outline of Fourier Series: Math 201B

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1 Functions and convolutions

1.1 Periodic functions

- **Periodic functions.** Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ denote the circle, or one-dimensional torus. A function $f : \mathbb{T} \to \mathbb{C}$ is equivalent to a 2π-periodic function $\tilde{f} : \mathbb{R} \to \mathbb{C}$. (The real line $\mathbb{R}$ is the universal cover of the circle $\mathbb{T}$ and $\tilde{f}$ is the lift of f from $\mathbb{T}$ to $\mathbb{R}$.) We identify f with $\tilde{f}$.

- **Continuous functions.** The space of continuous functions $f : \mathbb{T} \to \mathbb{C}$ is denoted by $C(\mathbb{T})$. It is a Banach space when equipped with the maximum (sup) norm.

- **Smooth functions.** If $k \in \mathbb{N}$, the space k-times continuously differentiable functions is denoted by $C^k(\mathbb{T})$. This is a Banach space with the C^k-norm (the sum of the maximum values of a function and its derivatives of order less than or equal to k.) The space of smooth functions (functions with continuous derivatives of all orders) is denoted by $C^\infty(\mathbb{T})$. This is a Fréchet space with the metric

$$d(\phi, \psi) = \sum_{k=0}^{\infty} \frac{1}{2^k} \left(\frac{\|\phi - \psi\|_{C^k}}{1 + \|\phi - \psi\|_{C^k}} \right).$$

- **L^p-spaces.** For $1 \le p < \infty$, the Banach space $L^p(\mathbb{T})$ consists of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ such that

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

  The space $L^\infty(\mathbb{T})$ consists of essentially bounded functions. We identify functions that are equal almost everywhere.

- **L^2-Hilbert space.** The space $L^2(\mathbb{T})$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{\mathbb{T}} \overline{f(x)} g(x)\, dx$$

- **Density.** The space $C^\infty(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ for $1 \leq p < \infty$ and in $C(\mathbb{T})$, but it is not dense in $L^\infty(\mathbb{T})$. More specifically, according to the Weierstrass approximation theorem, the same density results are true for the space $\mathcal{P}(\mathbb{T})$ of trigonometric polynomials of the form $\sum_{|n|\leq N} c_n e^{inx}$.

1.2 Convolutions and approximate identities

- **Convolution.** If $f, g \in L^1(\mathbb{T})$, the convolution $f * g \in L^1(\mathbb{T})$ is defined by

$$(f * g)(x) = \int_{\mathbb{T}} f(x-y) g(y)\, dy.$$

- **Young's inequality.** If $1 \leq p, q, r \leq \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$$

  and $f \in L^p(\mathbb{T})$, $g \in L^q(\mathbb{T})$, then $f * g \in L^r(\mathbb{T})$. Moreover,

$$\|f * g\|_r \leq \|f\|_p \|g\|_q .$$

  This follows from Fubini's theorem. In particular, convolution with an L^1-function is a bounded map on L^p,

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p ;$$

  and the convolution of L^2-functions is bounded (and therefore continuous by a density argument)

$$\|f * g\|_\infty \leq \|f\|_2 \|g\|_2 .$$

- **Approximate identity.** A sequence of functions

$$\{\phi_n \in L^1(\mathbb{T}) : n \in \mathbb{N}\}$$

is an approximate identity if there exists a constant M such that

$$\int_{\mathbb{T}} \phi_n \, dx = 1 \qquad \text{for every } n,$$

$$\int_{\mathbb{T}} |\phi_n| \, dx \le M \qquad \text{for all } n,$$

$$\lim_{n\to\infty} \int_{\delta<|x|<\pi} |\phi_n| \, dx = 0 \qquad \text{for every } \delta > 0.$$

(Analogous definitions apply to a family of functions that depend on a continuous parameter.)

- **Mollification.** If $\{\phi_n\}$ is an approximate identity and $f \in C(\mathbb{T})$ then $\phi_n * f \in C(\mathbb{T})$ and $\phi_n * f \to f$ uniformly as $n \to \infty$. If $f \in L^p(\mathbb{T})$, and $1 \le p < \infty$, then $\phi_n * f \to f$ in L^p as $n \to \infty$. If $\phi_n \in C^\infty(\mathbb{T})$ and $f \in L^1(\mathbb{T})$, then $\phi_n * f \in C^\infty(\mathbb{T})$. (The Lebesgue dominated convergence theorem justifies "differentiating under the integral sign.")

2 Fourier Series

2.1 L^1-theory

- **Definition of Fourier coefficients.** For $f \in L^1(\mathbb{T})$ define the Fourier coefficients $\hat{f} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-inx} \, dx.$$

- **Fourier coefficients determine a function.** If $f, g \in L^1(\mathbb{T})$ and

$$\hat{f}(n) = \hat{g}(n) \qquad \text{for all } n \in \mathbb{Z}$$

then $f = g$ (up to pointwise a.e.-equivalence). Thus follows from approximation of a function by convolution with an approximate identity that consists of trigonometric polynomials (*e.g.* the Féjer kernel.)

- **Riemann-Lebesgue lemma.** If $f \in L^1(\mathbb{T})$, then

$$\hat{f}(n) \to 0 \qquad \text{as } |n| \to \infty.$$

This result follows from the estimate

$$\left\| \hat{f} \right\|_{\ell^\infty} \le \frac{1}{2\pi} \|f\|_{L^1}$$

and the density of trigonometric polynomials (or smooth functions) in $L^1(\mathbb{T})$.

- **Convolution theorem.** If $f, g \in L^1(\mathbb{T})$, then

$$\widehat{(f * g)}(n) = 2\pi \hat{f}(n)\hat{g}(n).$$

That is, the Fourier transform maps the convolution product of functions to the pointwise product of their Fourier coefficients.

2.2 L^2-theory

- **Fourier basis.** The functions

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z} \right\}$$

form an orthonormal basis of $L^2(\mathbb{T})$. The orthonormality is easy to verify; the completeness follow by the use of convolution with an approximate identity that consists of trigonometric polynomials to approximate a general $f \in L^2(\mathbb{T})$.

- **Fourier series of an L^2-function.** A function $f \in L^1(\mathbb{T})$ belongs to $L^2(\mathbb{T})$ if and only if

$$\sum_{n \in \mathbb{Z}} \left| \hat{f}(n) \right|^2 < \infty$$

and then (with the above normalization of the Fourier coefficients)

$$f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx}$$

where the series converges unconditionally with respect to the L^2-norm.

- **Parseval's theorem.** If $f, g \in L^2(\mathbb{T})$ then

$$\langle f, g \rangle = 2\pi \sum_{n \in \mathbb{Z}} \overline{\hat{f}(n)} \hat{g}(n).$$

2.3 Absolutely convergent Fourier series

- **Absolutely convergent Fourier series.** If $f \in L^1(\mathbb{T})$ has absolutely convergent Fourier coefficients $\hat{f} \in \ell^1(\mathbb{Z})$, meaning that

$$\sum_{n\in\mathbb{Z}} \left|\hat{f}(n)\right| < \infty,$$

  then $f \in C(\mathbb{T})$. This follows from the fact that the Fourier series of f converges uniformly to f by the Weierstrass M-test. We denote the space of functions with absolutely convergent Fourier series by $A(\mathbb{T})$.

- **Pointwise divergence of Fourier series.** There are continuous functions whose Fourier series converge uniformly but not absolutely, and continuous function whose Fourier series do not do not converge uniformly; in fact, there are continuous functions whose Fourier series diverge pointwise on an arbitrary set of Lebesgue measure zero. If $f \in L^p(\mathbb{T})$ for $1 < p \leq \infty$, then the Fourier series of f converges pointwise a.e. to f (Carlson, Hunt) but there exist functions $f \in L^1(\mathbb{T})$ whose Fourier series diverge pointwise a.e. (Kolmogorov).

- **Convolution theorem.** If $f, g \in A(\mathbb{T})$, then $fg \in A(\mathbb{T})$ and

$$\widehat{(fg)}(n) = 2\pi \sum_{k\in\mathbb{Z}} \hat{f}(n-k)\hat{g}(k).$$

  That is, the Fourier transform maps the pointwise product of functions to the discrete convolution product of their Fourier coefficients.

2.4 Weak derivatives and Sobolev spaces

- **Weak derivative.** A function $f \in L^1(\mathbb{T})$ has weak derivative $g = f' \in L^1(\mathbb{T})$ if

$$\int_{\mathbb{T}} f\phi'\, dx = -\int_{\mathbb{T}} g\phi\, dx \qquad \text{for all } \phi \in C^\infty(\mathbb{T}).$$

  That is, weak derivatives are defined by integration by parts. If $f \in C^1(\mathbb{T})$, then the weak derivative agrees with the pointwise derivative (up to pointwise a.e.-equivalence).

- **Fourier coefficients of derivatives.** If $f \in L^1(\mathbb{T})$ has weak derivative $f' \in L^1(\mathbb{T})$, then

$$\widehat{f'}(n) = in\hat{f}(n).$$

- **Decay of Fourier coefficients.** If $k \in \mathbb{N}$ and $f \in W^{k,1}(\mathbb{T})$, meaning that f has weak derivatives $f', f'', \dots, f^{(k)} \in L^1(\mathbb{T})$ of order less than or equal to k, then

$$|n|^k \hat{f}(n) \to 0 \qquad \text{as } |n| \to \infty.$$

This follows from an application of the Riemann-Lebesgue lemma to $f^{(k)}$.

- **L^2-Sobolev spaces.** If $0 \le s < \infty$, the Sobolev space $H^s(\mathbb{T})$ consists of all functions $f \in L^2(\mathbb{T})$ such that

$$\sum_{n \in \mathbb{Z}} \left(1 + n^2\right)^s \left|\hat{f}(n)\right|^2 < \infty.$$

This is a Hilbert space with inner product

$$\langle f, g \rangle_{H^s} = 2\pi \sum_{n \in \mathbb{Z}} \left(1 + n^2\right)^s \overline{\hat{f}(n)} \hat{g}(n).$$

If $k \in \mathbb{N}$ is an integer, then $H^k(\mathbb{T})$ consists of all functions $f \in L^2(\mathbb{T})$ that have weak derivatives of order less than or equal to k belonging to $L^2(\mathbb{T})$.

- **Sobolev embedding.** If $f \in H^s(\mathbb{T})$ and $s > 1/2$, then $f \in C(\mathbb{T})$ and there is a constant $C > 0$, depending only on s, such that

$$\|f\|_\infty \le C \|f\|_{H^s} \qquad \text{for all } f \in H^s(\mathbb{T}).$$

This theorem follows by showing that $\hat{f} \in \ell^1(\mathbb{Z})$ is absolutely convergent so $f \in A(\mathbb{T})$. Roughly speaking, for functions of a single variable, more than one-half an L^2-derivative implies continuity.

2.5 Periodic distributions

- **Test functions.** The space $\mathcal{D}(\mathbb{T})$ of periodic test functions consists of all smooth functions $\phi \in C^\infty(\mathbb{T})$ with the following notion of convergence of test functions: $\phi_n \to \phi$ in $\mathcal{D}(\mathbb{T})$ if

$$\phi_n^{(k)} \to \phi^{(k)} \qquad \text{uniformly as } n \to \infty \text{ for every } k = 0, 1, 2, \dots.$$

Here $\phi^{(k)}$ denotes the kth derivative of ϕ.

- **Fourier series of test functions.** A function $\phi \in L^1(\mathbb{T})$ belongs to $\mathcal{D}(\mathbb{T})$ if and only if its Fourier coefficients are rapidly decreasing,

$$|n|^k \hat{\phi}(n) \to 0 \qquad \text{as } |n| \to \infty \text{ for every } k \in \mathbb{N}.$$

This follows from the decay estimates for the Fourier coefficients of smooth functions and the Sobolev embedding theorem. (Note that $\phi \in H^k$ for every $k \in \mathbb{N}$ if and only if $\phi \in C^k$ for every $k \in \mathbb{N}$.) The Fourier series of $\phi \in \mathcal{D}(\mathbb{T})$ converges to ϕ in the sense of test functions.

- **Distributions.** A distribution T is a continuous linear functional

$$T : \mathcal{D}(\mathbb{T}) \to \mathbb{C}.$$

The space of distributions is denoted $\mathcal{D}'(\mathbb{T})$ and the action of $T \in \mathcal{D}'$ on $\phi \in \mathcal{D}$ by $\langle T, \phi \rangle$. (This duality paring is linear in both arguments, not anti-linear in the first argument like the inner product on a Hilbert space.)

- **Convergence of distributions.** A sequence of of distributions $\{T_n\}$ converges to a distribution T, written $T_n \rightharpoonup T$, if

$$\langle T_n, \phi \rangle \to \langle T, \phi \rangle \qquad \text{as } n \to \infty \text{ for every } \phi \in \mathcal{D}(\mathbb{T}).$$

- **Order of a distribution.** If $T \in \mathcal{D}'(\mathbb{T})$, there is a non-negative integer k and a constant C such that

$$|\langle T, \phi \rangle| \leq C \|\phi\|_{C^k} \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{T}).$$

The minimal such integer k is called the order of T.

- **Regular distributions.** If $f \in L^1(\mathbb{T})$, we define $T_f \in \mathcal{D}'(\mathbb{T})$ by

$$\langle T_f, \phi \rangle = \int_{\mathbb{T}} f \phi \, dx.$$

Any distribution of this form is called a regular distribution. We identify f with T_f and regard $L^1(\mathbb{T})$ as a subspace of $\mathcal{D}'(\mathbb{T})$.

- **Distributional derivative.** Every $T \in \mathcal{D}(\mathbb{T})$ has a distributional derivative $T' \in \mathcal{D}(\mathbb{T})$ defined by

$$\langle T', \phi \rangle = -\langle T, \phi' \rangle$$

A function $f \in L^1(\mathbb{T})$ has a weak derivative $f' \in L^1(\mathbb{T})$ if and only if its distributional derivative is regular, and then $(T_f)' = T_{f'}$.

- **Fourier series of distributions.** The Fourier coefficients $\hat{T} : \mathbb{Z} \to \mathbb{C}$ of a distribution $T \in \mathcal{D}(\mathbb{T})$ are defined by

$$\hat{T}(n) = \frac{1}{2\pi} \langle T, e^{-inx} \rangle$$

A linear functional on $\mathcal{D}(\mathbb{T})$ is a distribution if and only if its Fourier coefficients have slow growth, meaning that the exists a non-negative integer k and a constant C such that

$$\left| \hat{T}(n) \right| \leq C \left(1 + n^2\right)^{k/2} \qquad \text{for all } n \in \mathbb{Z}.$$

In that case, the Fourier series of T,

$$\sum_{n \in \mathbb{Z}} \hat{T}(n) e^{inx}$$

converges to T in the sense of distributions.

- **The delta function.** The periodic δ-function supported at 0 is the distribution $\delta \in \mathcal{D}'(\mathbb{T})$ defined by

$$\langle \delta, \phi \rangle = \phi(0)$$

This is a distribution of order zero, but it is not a regular distribution. (It is, in fact, a measure.) Its Fourier series is

$$\delta = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{inx}.$$