Outline of Bounded Operators: Math 201B

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1 Linear functionals on Hilbert spaces

- **Linear functionals.** A bounded linear function on a complex Hilbert space $\mathcal{H}$ is a bounded scalar-valued linear map $\phi : \mathcal{H} \to \mathbb{C}$. (We replace $\mathbb{C}$ by $\mathbb{R}$ for real spaces.)

- **Dual space.** The space of bounded linear functionals on $\mathcal{H}$ is the topological dual space of $\mathcal{H}$, denoted $\mathcal{H}^*$. The norm of $\phi : \mathcal{H} \to \mathbb{C}$ is

$$\|\phi\|_{\mathcal{H}^*} = \sup_{x \neq 0} \left(\frac{|\phi(x)|}{\|x\|} \right) = \sup_{\|x\|=1} |\phi(x)|$$

- **Reisz representation theorem.** If $\phi \in \mathcal{H}^*$ then there is a unique $x \in \mathcal{H}$ such that

$$\phi(y) = \langle x, y \rangle \qquad \text{for every } y \in \mathcal{H}.$$

  The mapping $J : \mathcal{H}^* \to \mathcal{H}$ defined by $J : \phi \mapsto x$ is a conjugate-linear (*i.e.* $J(\lambda\phi) = \bar{\lambda} J\phi$) isometric isomorphism of $\mathcal{H}^*$ onto $\mathcal{H}$. Thus, using J, we may identify the dual space of a Hilbert space with the Hilbert space itself.

- **Weak convergence.** A sequence $\{x_n\}$ in $\mathcal{H}$ converges weakly to $x \in \mathcal{H}$, written $x_n \rightharpoonup x$, if

$$\langle x_n, y \rangle \to \langle x, y \rangle \qquad \text{for every } y \in \mathcal{H}.$$

- **Norm properties of weak convergence.** If $x_n \rightharpoonup x$ as $n \to \infty$, then $\{\|x_n\| : n \in \mathbb{N}\}$ is bounded and

$$\|x\| \leq \liminf_{n \to \infty} \|x_n\|$$

i.e. the norm is weakly lower semi-continuous. If

$$x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\|$$

then $x_n \to x$ strongly (in norm).

- **Necessary and sufficient condition for weak convergence.** Let D be a dense subset of a Hilbert space $\mathcal{H}$. Then $x_n \rightharpoonup x$ in $\mathcal{H}$ if and only if $\{\|x_n\|\}$ is bounded and

$$\langle x_n, y\rangle \to \langle x, y\rangle \qquad \text{for every } y \in D.$$

- **Banach-Alaoglu theorem.** The closed unit ball of a Hilbert space is weakly compact.

- **Minimization problems.** Let D be a weakly closed subset of a Hilbert space $\mathcal{H}$. A real-valued function $F : D \subset \mathcal{H} \to \mathbb{R}$ is weakly lower semi-continuous (wlsc) on D if

$$F(x) \leq \liminf_{n\to\infty} F(x_n)$$

for all weakly convergent sequences $\{x_n\}$ in D, where $x_n \rightharpoonup x$ as $n \to \infty$. If D is weakly closed and bounded and F is wlsc on D, then F is bounded from below and attains its infimum on D.

2 Bounded linear operators on a Hilbert space

- **Bounded operators.** A linear operator $A : \mathcal{H} \to \mathcal{K}$ between Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ is bounded if its operator norm

$$\|A\| = \sup_{x\neq 0}\left(\frac{\|Ax\|_{\mathcal{K}}}{\|x\|_{\mathcal{H}}}\right) = \sup_{\|x\|_{\mathcal{H}}=1} \|Ax\|_{\mathcal{K}} = \sup_{\|x\|_{\mathcal{H}}=1, \|y\|_{\mathcal{K}}=1} |\langle Ax, y\rangle_{\mathcal{K}}|\,.$$

is finite. The space of bounded linear maps from $\mathcal{H}$, $\mathcal{K}$ is denoted $\mathcal{B}(\mathcal{H}, \mathcal{K})$. It is a Banach space when equipped with the operator norm. If $\mathcal{H} = \mathcal{K}$, we write $\mathcal{B}(\mathcal{H}, \mathcal{H}) = \mathcal{B}(\mathcal{H})$.

- **Adjoints.** The (Hilbert-space) adjoint of an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is the bounded operator $A^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$\langle Ax, y\rangle_{\mathcal{K}} = \langle x, A^* y\rangle_{\mathcal{H}} \qquad \text{for all } x \in \mathcal{H} \text{ and all } y \in \mathcal{K}.$$

- **The algebra $\mathcal{B}(\mathcal{H})$.** The Banach space $\mathcal{B}(\mathcal{H})$ is a C^*-algebra with respect to the composition product and the adjoint operation:

$$\|AB\| \leq \|A\|\|B\|, \qquad A^{**} = A, \qquad (AB)^* = B^*A^*.$$

  The commutator of $A, B \in \mathcal{B}(\mathcal{H})$ is the operator $[A, B] \in \mathcal{B}(\mathcal{H})$ defined by $[A, B] = AB - BA$.

- **Kernel-range theorem.** If $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a Hilbert space $\mathcal{H}$, then the kernel of A

$$\ker A = \{x \in \mathcal{H} : Ax = 0\}$$

  is a closed linear subspace of $\mathcal{H}$, and the range of A

$$\operatorname{ran} A = \{y \in \mathcal{H} : y = Ax \text{ for some } x \in \mathcal{H}\}$$

  is a linear subspace of $\mathcal{H}$, which may or may not be closed. We always have

$$\mathcal{H} = \overline{\operatorname{ran} A} \oplus \ker A^*.$$

- **Self-adjoint operators.** A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is self-adjoint if $A^* = A$, meaning that

$$\langle Ax, y\rangle = \langle x, Ay\rangle \qquad \text{for all } x, y \in \mathcal{H}.$$

- **Sesquilinear forms.** A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ defines a sesquilinear form $a : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ (meaning that a is conjugate-linear in the first argument and linear in the second argument) by

$$a(x, y) = \langle x, Ay\rangle.$$

  If A is self-adjoint, then $a(x, y) = \overline{a(y, x)}$, $a(x, x) \in \mathbb{R}$, and

$$\|A\| = \sup_{x \neq 0} \frac{|\langle x, Ax\rangle|}{\|x\|^2}.$$

- **Normal operators.** A bounded linear operator $A : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is normal if A^*, A commute, meaning that

$$A^*A = AA^*.$$

  Self-adjoint and unitary operators on $\mathcal{H}$ are normal.

- **Unitary operators.** An operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is unitary if

$$U^*U = I_{\mathcal{H}}, \qquad UU^* = I_{\mathcal{K}}.$$

In that case, U maps any orthonormal basis of $\mathcal{H}$ to an orthonormal basis of $\mathcal{K}$, and preserves inner-products,

$$\langle Ux, Uy \rangle_{\mathcal{K}} = \langle x, y \rangle_{\mathcal{H}} \qquad \text{for all } x, y \in \mathcal{H},$$

so U defines an isometric isomorphism of $\mathcal{H}$ onto $\mathcal{K}$.

- **Orthogonal projections.** An orthogonal projection on a Hilbert space $\mathcal{H}$ is a bounded linear operator $P \in \mathcal{B}(\mathcal{H})$ such that $P^2 = P$ (projection) and $P^* = P$ (self-adjoint or orthogonal).

- **Projection theorem.** Every orthogonal projection P on $\mathcal{H}$ gives a direct sum decomposition

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}, \qquad \mathcal{M} = \operatorname{ran} P, \quad \mathcal{M}^{\perp} = \ker P$$

where $\mathcal{M}$ is a closed linear subspace of $\mathcal{H}$. Conversely, every closed subspace $\mathcal{M} \subset \mathcal{H}$ is associated with an orthogonal projection in this way.