Brief Notes on Measure Theory

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Measure Spaces

Suppose that X is a non-empty set. A collection $\mathcal{A}$ of subsets of X is called a *σ-algebra* on X if it contains X and is closed under the operations of taking complements, countable unions, and countable intersections:

- $\emptyset, X \in \mathcal{A}$;
- if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$;
- if $E_n \in \mathcal{A}$ for $n \in \mathbb{N}$, then

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}, \qquad \bigcap_{n=1}^{\infty} E_n \in \mathcal{A}.$$

Of course, the fact that $\mathcal{A}$ is closed under countable intersections is implied by the fact that it is closed under complements and countable unions.

If X is a set and $\mathcal{A}$ is a σ-algebra on X, then we call $(X, \mathcal{A})$ a *measurable space*, and elements of $\mathcal{A}$ are called *measurable sets*.

1 Example If X is any nonempty set, then the largest σ-algebra on X is the power set $\mathcal{P}(X)$ consisting of all subsets of X. The smallest σ-algebra is $\{\emptyset, X\}$.

If $\mathcal{F}$ is any collection of subsets of X, then the smallest σ-algebra containing $\mathcal{F}$ is called the *σ-algebra generated by* $\mathcal{F}$. This σ-algebra is the intersection of all σ-algebras that contain $\mathcal{F}$.

2 Example If $X = \mathbb{R}$ is the set of real numbers and $\mathcal{G}$ is the collection of open sets, then the σ-algebra generated by $\mathcal{G}$ is called the *Borel σ-algebra* on $\mathbb{R}$, which we denote by $\mathcal{B}(\mathbb{R})$. Elements of $\mathcal{B}(\mathbb{R})$ are called *Borel* sets. The σ-algebra $\mathcal{B}(\mathbb{R})$ is also generated by the collection $\mathcal{F}$ of closed sets, or by various collection of intervals, such as $\{(a, \infty) \mid a \in \mathbb{R}\}$. We define the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ on $\mathbb{R}^d$ in a similar way. More generally, if X is any topological space, then the Borel σ-algebra $\mathcal{B}(X)$ is the σ-algebra generated by the open sets of X.

3 Remark It is surprisingly complicated to obtain $\mathcal{B}(\mathbb{R})$ by starting from the open sets and taking successively complements, countable unions, and countable intersections. This process requires uncountably many iterations before it gives $\mathcal{B}(\mathbb{R})$.

If $(X, \mathcal{A})$ is a measurable space, then a map

$$\mu : \mathcal{A} \to [0, \infty]$$

is called a *measure* if:

- $\mu(\emptyset) = 0$;
- if $\{E_n \mid n \in \mathbb{N}\}$ is a collection of disjoint sets in $\mathcal{A}$, meaning that $E_m \cap E_n = \emptyset$ for $m \neq n$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu\left(E_n\right).$$

  Here, we use the obvious conventions for measures and sums that are equal to ∞.

The second property in the definition of a measure is called 'countable additivity', and it expresses the idea that the 'volume' of a disjoint union should be the sum of the 'volumes' of its individual parts. We call $(X, \mathcal{A}, \mu)$ a *measure space*.

It follows from the definition that: (a) if $E \subset F$ are measurable sets, then $\mu(E) \leq \mu(F)$; (b) if $E_1 \subset E_2 \subset E_3 \subset \ldots$ is an increasing sequence of measurable sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n);$$

(c) if $E_1 \supset E_2 \supset E_3 \supset \ldots$ is a decreasing sequence of measurable sets *with finite measure for some* $n \in \mathbb{N}$ (see the following example), then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n).$$

4 Example Let X be any set. Define $\nu : \mathcal{P}(X) \to [0, \infty]$ by

$$\nu(E) = \#\text{ points in } E.$$

Then ν is a measure on $\mathcal{P}(X)$ called *counting measure*. If $X = \mathbb{N}$ and

$$E_n = \{1, 2, 3, \ldots, n\},$$

then (E_n) is an increasing sequence of sets whose union is $\mathbb{N}$. We have $\nu(E_n) = n$, $\nu(\mathbb{N}) = \infty$, and $\nu(E_n) \to \nu(\mathbb{N})$ as $n \to \infty$. On the other hand, if

$$E_n = \{n, n+1, n+2, \ldots\},$$

then (E_n) is an decreasing sequence of sets whose intersection is $\emptyset$. We have $\nu(E_n) = \infty$, $\nu(\emptyset) = 0$, and $\nu(E_n) \not\to \nu(\emptyset)$ as $n \to \infty$.

5 Example There is a unique measure $m : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ on $\mathbb{R}$, called *Lebesgue measure*, such that

$$m(I) = \text{Length}(I)$$

for any interval $I \subset \mathbb{R}$. Here, the length of an interval is defined in the usual way; for example, $\text{Length}((a, b]) = b - a$. More generally, there is a unique Lebesgue measure $m : \mathcal{B}(\mathbb{R}^d) \to [0, \infty]$ on $\mathbb{R}^d$ such that

$$m(I) = \text{Volume}(I)$$

for any d-dimensional rectangle $I \subset \mathbb{R}^d$. By a rectangle I in $\mathbb{R}^d$, we mean a set of the form $I = I_1 \times I_2 \times, \ldots \times I_d$, where the $I_k \subset \mathbb{R}$ are intervals, and the volume of I is the product of the lengths of its sides I_k. The Lebesgue measure is of a set is invariant under translations and rotations.

We say that a measure space $(X, \mathcal{A}, \mu)$ is *finite* if the set X has finite measure, and *σ-finite* if X is a countable union of sets of finite measure. For example, any probablity space, which is a measure space such that $\mu(X) = 1$ is finite, and $\mathbb{R}^d$ equipped with Lebesgue measure is σ-finite but not finite.

A set E has measure zero if $E \in \mathcal{A}$ and $\mu(E) = 0$. A property that holds except on a set of measure zero is said to hold *almost everywhere* (or a.e. for short). For example, we say that two functions $f, g : X \to Y$ are equal a.e. if $\mu\{x \in X \mid f(x) \neq g(x)\} = 0$.

6 Example The Lebesgue measure of any singleton $\{a\} \subset \mathbb{R}$ is zero. It follows from the countable additivity of a measure that the Lebesgue measure of any countable set is zero; for example, the set of rationals has Lebesgue measure zero. There are, however, many uncountable sets whose Lebesgue measure is zero; for example, the standard Cantor set has Lebesgue measure zero. In general, the Lebesgue measure of a subset $N \subset \mathbb{R}^d$ is zero if and only if for every $\epsilon > 0$ there is a countable collection $\{I_n \mid n \in \mathbb{N}\}$ of, not necessarily disjoint, d-dimensional rectangles such that

$$N \subset \bigcup_{n=1}^{\infty} I_n, \qquad \sum_{n=1}^{\infty} \text{Volume}(I_n) < \epsilon.$$

A measure space $(X, \mathcal{A}, \mu)$ is *complete* if every subset of a set of measure zero is measurable (when its measure is necessarily zero). Every measure space $(X, \mathcal{A}, \mu)$ has a unique completion $(X, \overline{\mathcal{A}}, \overline{\mu})$, which is the smallest complete measure space such that $\overline{\mathcal{A}} \supset \mathcal{A}$ and $\overline{\mu}|_{\mathcal{A}} = \mu$.

7 Example Lebesgue measure on the Borel σ-algebra $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ is not complete, meaning that there are Borel sets of Lebesgue measure zero which contain subsets that are not Borel sets. The completion of the Borel σ-algebra with respect to Lebesgue measure is the σ-algebra $\mathcal{L}(\mathbb{R})$ of *Lebesgue measurable* sets. Similarly, the σ-algebra $\mathcal{L}(\mathbb{R}^d)$ of Lebesgue measurable sets in $\mathbb{R}^d$ is the completion of the σ-algebra $\mathcal{B}(\mathbb{R}^d)$ of Borel measurable sets with respect to Lebesgue measure.

8 Remark It is not possible to extend Lebesgue measure to a countably additive measure defined on *all* subsets of $\mathbb{R}$. To 'construct' a nonmeasurable set, let $x \sim y$ be the equivalence relation on $\mathbb{R}$ defined by $x - y \in \mathbb{Q}$. Then $\mathbb{R}$ is a disjoint union of uncountably many equivalence classes of $\sim$, each of which contains countably many elements. Using the axiom of choice, we pick an uncountable set $E \subset \mathbb{R}$ containing exactly one element from each equivalence class of $\sim$. One can show that E is not Lebesgue measurable. (This follows from the fact that Lebesgue measure is translation invariant and countably additive, and $\mathbb{R}$ is a countable disjoint union of translates of E by rational numbers.) Solovay (1970) proved that the axiom of choice is required to show the existence of a set which is not Lebesgue measurable.

9 Remark In $\mathbb{R}^d$ with $d \geq 3$ it is possible to decompose a ball (using the axiom of choice) into a finite number of (non-Lebesgue measurable) subsets and reassemble these pieces into two balls of the same volume. This Banach-Tarski paradox shows that in $d \geq 3$ space-dimensions it is not even possible to define a finitely-additive translation and rotation invariant set function on all subsets of $\mathbb{R}^d$ whose value on any ball is equal to its volume. There exist finitely-additive translation and rotation invariant measures on $\mathbb{R}^2$, but they still have some counter-intuitive properties. For example, Laczkovich (1990) proved that it is possible to decompose a circular disc into a finite number of pieces (in Laczkovich's proof, approximately 10^{50} non-Lebesgue measurable sets) and reassemble them into a square of the same area.

Measurable Functions

If $(X, \mathcal{A})$, $(Y, \mathcal{B})$ are measurable spaces, we say that a function $f : X \to Y$ is *measurable* if $f^{-1}(E) \in \mathcal{A}$ for every $E \in \mathcal{B}$.

If $\mathcal{B}$ is generated by $\mathcal{F}$ then f is measurable if and only if $f^{-1}(E) \in \mathcal{A}$ for every $E \in \mathcal{F}$. For example, if X, Y are topological spaces equipped with their Borel σ-algebras, then every continuous function $f : X \to Y$ is measurable.

In the case when $Y = \mathbb{R}$ and

$$f : X \to \mathbb{R},$$

we equip $\mathbb{R}$ with its Borel σ-algebra $\mathcal{B}(\mathbb{R})$. Then f is measurable if and only if $f^{-1}\left((a, \infty)\right) \in \mathcal{A}$ for every $a \in \mathbb{R}$. A function

$$f : X \to \mathbb{C}$$

is measurable if and only if its real and imaginary parts $\Re f$ and $\Im f$ are measurable.

It is often convenient to consider extended real-valued functions

$$f : X \to \overline{\mathbb{R}},$$

where $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$, with the appropriate Borel σ-algebra. Then f is measurable if and only if $f^{-1}\left((a, \infty]\right)$ is measurable for every $a \in \overline{\mathbb{R}}$.

10 Example The *characteristic function* (or *indicator function*) of a subset $E \subset X$ is the function $\chi_E : X \to \mathbb{R}$ defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Then χ_E is a measurable function if and only if E is a measurable set.

A function $\phi : X \to \mathbb{R}$ is a *simple function* if

$$\phi(x) = \sum_{n=1}^{N} c_n \chi_{E_n}(x)$$

for some $c_1, \dots, c_N \in \mathbb{R}$ and $E_1, \dots, E_N \in \mathcal{A}$. Note that, according to this definition, a simple function is measurable.

We say that a sequence of functions $f_n : X \to \overline{\mathbb{R}}$ *converges pointwise* to a function $f : X \to \overline{\mathbb{R}}$ if $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$. (Here, we use the obvious conventions about sequences that converge to $\pm\infty$.) We say that $f_n \to f$ pointwise a.e. if $f_n(x) \to f(x)$ as $n \to \infty$ except for $x \in N$ where $N \subset X$ is a set of measure zero.

11 Theorem If $f_n : X \to \overline{\mathbb{R}}$ are measurable functions and $f_n \to f$ pointwise as $n \to \infty$, then $f : X \to \overline{\mathbb{R}}$ is measurable. If X is a complete measure space and $f_n \to f$ pointwise a.e. as $n \to \infty$, then $f : X \to \overline{\mathbb{R}}$ is measurable.

Integration

If $\phi : X \to [0, \infty)$ is a nonnegative simple function on a measure space $(X, \mathcal{A}, \mu)$, given by $\phi = \sum_{n=1}^{N} c_n \chi_{E_n}$ where $0 \le c_n < \infty$ and $E_n \in \mathcal{A}$, then we define the integral of ϕ with respect to the μ by

$$\int \phi \, d\mu = \sum_{n=1}^{N} c_n \mu\left(E_n\right).$$

We use the convention that $0 \cdot \infty = 0$, meaning that the integral of a function that is 0 on a set of measure ∞, or ∞ on a set of measure 0, is equal to 0.

12 Example The characteristic function $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ of the rationals is not Riemann integrable on any compact interval of non-zero length, but it is Lebesgue integrable with

$$\int \chi_{\mathbb{Q}} \, dm = 1 \cdot m(\mathbb{Q}) = 0.$$

If $f : X \to [0, \infty]$ is a nonnegative, measurable, extended real-valued function, we define

$$\int f \, d\mu = \sup \left\{ \int \phi \, d\mu \mid 0 \le \phi \le f, \, \phi \text{ simple} \right\}.$$

13 Remark In this definition, we approximate the function f from below by simple functions. In contrast with the definition of the Riemann integral, it is not necessary to approximate a measurable function from both above and below in order to define its integral.

If $f : X \to \overline{\mathbb{R}}$, we write $f = f_+ - f_-$ in terms of its positive and negative parts

$$f_+ = \max\{f, 0\}, \qquad f_- = \max\{-f, 0\},$$

and define

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu,$$

provided that at least one of the integrals $\int f_+ \, d\mu$, $\int f_- \, d\mu$ is finite.

We also have

$$\int |f| \, d\mu = \int f_+ \, d\mu + \int f_- \, d\mu.$$

We say that $f : X \to \overline{\mathbb{R}}$ is *integrable* if

$$\int |f| \, d\mu < \infty,$$

which occurs if and only if both $\int f_+ \, d\mu$, $\int f_- \, d\mu$ are finite.

If $f : X \to \mathbb{C}$ is a complex valued function $f = g+ih$, then f is measurable if and only if its real and imaginary parts $g, h : X \to \mathbb{R}$ are measurable, and integrable if and only if g, h are integrable. In that case, we define

$$\int f \, d\mu = \int g \, d\mu + i \int h \, d\mu.$$

If $A \subset X$ is measurable, then we define

$$\int_A f \, d\mu = \int f \chi_A \, d\mu.$$

In contrast with the Riemann integral, where integrals over non-rectangular subsets of $\mathbb{R}^2$ already present problems, it is trivial to define the Lebesgue integral over arbitrary measurable subsets.

This Lebesgue integral has all the usual properties of an integral. For example, if $f, g : X \to \mathbb{R}$ and $f \leq g$ then

$$\int f \, d\mu \leq \int g \, d\mu,$$

and if $f, g : X \to \mathbb{C}$, $\lambda \in \mathbb{C}$ then

$$\begin{aligned} \left| \int f \, d\mu \right| &\le \int |f| \, d\mu, \\ \int \lambda f \, d\mu &= \lambda \int f \, d\mu, \\ \int (f + g) \, d\mu &= \int f \, d\mu + \int g \, d\mu. \end{aligned}$$

14 Example Suppose that $X = \mathbb{N}$ and $\nu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is counting measure on $\mathbb{N}$. If $f : \mathbb{N} \to \mathbb{R}$ and $f(n) = x_n$, then

$$\int_{\mathbb{N}} f \, d\nu = \sum_{n=1}^{\infty} x_n,$$

where the integral converges if and only if the series is absolutely convergent. Thus, the theory of absolutely convergent series is a special case of the Lebesgue integral. Note that conditionally convergent series, such as the alternating harmonic series, do not correspond to a Lebesgue integral, since both their positive and negative parts diverge.

15 Example Any Riemann integrable function $f : [a, b] \to \mathbb{R}$ is integrable with respect to Lebesgue measure, and the Riemann integral is equal to the Lebesgue integral,

$$\int_a^b f(x) \, dx = \int_{[a,b]} f \, dm.$$

Thus, all of the usual results from elementary calculus remain valid for the Lebesgue integral on $\mathbb{R}$. A Lebesgue integrable function, however, need not be Riemann integrable. In fact, a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is Lebesgue measurable and the set of discontinuities $\{x \in [a, b] \mid f \text{ is discontinuous at } x\}$ has Lebesgue measure zero. For example, the characteristic function of a Cantor set with non-zero measure is Lebesgue integrable, but it is not Riemann integrable, nor is any modification of the function on a set of measure zero Riemann integrable. We will often write an integral with respect to Lebesgue measure on $\mathbb{R}$ or $\mathbb{R}^d$ as

$$\int f \, dx.$$

Convergence Theorems

One of the most basic questions in integration theory is the following: If $f_n \to f$ pointwise, when can one say that

$$\int f_n \, d\mu \to \int f \, d\mu?$$

Some condition is necessary to ensure the convergence of the integrals, as can be seen from very simple examples.

16 Example Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \to 0$ as $n \to \infty$ pointwise on $\mathbb{R}$, but

$$\int f_n \, dx = 1 \qquad \text{for every } n \in \mathbb{N}.$$

The Riemann integral is not sufficiently general to permit a satisfactory answer to this question. For example, one can construct a monotone increasing sequence of nonnegative continuous functions $f_n : [0,1] \to \mathbb{R}$ that is bounded from above by 1 but whose pointwise limit is not even Riemann integrable.

The Lebsegue integral allows us to formulate simple and widely applicable conditions for the convergence of the integrals. The most important of these are that the pointwise convergence is monotone (the Monotone Convergence Theorem), or that every function f_n in the sequence is bounded by the same integrable function (the Lebesgue Dominated Convergence Theorem).

17 Theorem [Monotone Convergence] Suppose that (f_n) is an increasing sequence of nonnegative measurable functions $f_n : X \to [0,\infty]$ converging pointwise to $f : X \to [0,\infty]$, meaning that $f_n(x) \leq f_{n+1}(x)$ for every $n \in \mathbb{N}$ and $x \in X$. Then

$$\int f_n \, d\mu \to \int f \, d\mu \quad \text{as } n \to \infty.$$

Here, we use the obvious conventions about ∞.

18 Theorem [Dominated Convergence] Suppose that (f_n) is an sequence of measurable functions $f_n : X \to \mathbb{C}$ such that $f_n \to f$ pointwise and $|f_n| \leq g$ where $g : X \to [0, \infty]$ is an integrable function, meaning that

$$\int g\, d\mu < \infty.$$

Then

$$\int f_n\, d\mu \to \int f\, d\mu \quad \text{as } n \to \infty.$$

Here, all the integrals are necessarily finite.

Product Measures

Fubini's theorem provides a simple and general condition for the equality of multiple and iterated integrals of a function: namely, the function should be integrable (see Theorem 23 for multiple integrals on $\mathbb{R}^d$).

Suppose that $(X, \mathcal{A})$, $(Y, \mathcal{B})$ are measurable spaces. The *product σ-algebra* on the Cartesian product $X \times Y$ is the sigma algebra $\mathcal{A} \otimes \mathcal{B}$ generated by the collection of all measurable rectangles $\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Note that this collection is not itself a σ-algebra; for example, the union of two rectangles is not in general another rectangle.

19 Example Consider $\mathbb{R}^m$, $\mathbb{R}^n$ equipped with the Borel σ-algebras $\mathcal{B}(\mathbb{R}^m)$, $\mathcal{B}(\mathbb{R}^n)$. Then $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$, and one can show that

$$\mathcal{B}(\mathbb{R}^{m+n}) = \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R}^n).$$

Thus, the Borel σ-algebra on $\mathbb{R}^d$ can be obtained as a d-fold product of the Borel σ-algebra on $\mathbb{R}$. Unfortunately, this result does not remain quite true for Lebesgue measurable sets because the product of complete σ-algebras is not necessarily complete. For example, if $N \subset \mathbb{R}$ is not a Borel measurable subset of $\mathbb{R}$, then $\{0\} \times N$ is not a Borel-measurable subset of $\mathbb{R}^2$ even though it is contained in the subset $\{0\| \times \mathbb{R}$ of zero Lebesgue measure on $\mathbb{R}^2$. Instead on can show that the Lebesgue σ-algebra on $\mathbb{R}^{m+n}$ is the completion with respect to Lebesgue measure of the product of the Lebesgue σ-algebras on $\mathbb{R}^m$ and $\mathbb{R}^n$:

$$\mathcal{L}(\mathbb{R}^{m+n}) = \overline{\mathcal{L}(\mathbb{R}^m) \otimes \mathcal{L}(\mathbb{R}^n)}.$$

19 Theorem Suppose that $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces. There is a unique measure $\mu \otimes \nu : \mathcal{A} \otimes \mathcal{B} \to [0, \infty]$ on $X \times Y$ such that

$$(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B) \qquad \text{for all } A \in \mathcal{A},\ B \in \mathcal{B}.$$

If $f : X \times Y \to \mathbb{C}$ is a function of $(x, y) \in X \times Y$, then for each $x \in X$ we define the x-section $f_x : Y \to \mathbb{C}$ and for each $y \in Y$ we define the y-section $f^y : Y \to \mathbb{C}$ by

$$f_x(y) = f(x, y), \qquad f^y(x) = f(x, y).$$

20 Theorem If $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ are σ-finite measure spaces and $f : X \times Y \to \mathbb{C}$ is a measurable function, then $f_x : Y \to \mathbb{C}$, $f^y : X \to \mathbb{C}$ are measurable for every $x \in X$, $y \in Y$. Moreover, the functions $g : X \to \mathbb{C}$, $h : Y \to \mathbb{C}$ defined by

$$g(x) = \int f_x \, d\nu, \qquad h(y) = \int f^y \, d\mu$$

are measurable.

21 Theorem [Fubini] A measurable function $f : X \times Y \to \mathbb{C}$ is integrable if and only if either one of the iterated integrals

$$\int \left(\int |f^y| \, d\mu \right) d\nu, \qquad \int \left(\int |f_x| \, d\nu \right) d\mu$$

is finite. In that case

$$\int f \, d\mu \otimes d\nu = \int \left(\int f^y \, d\mu \right) d\nu = \int \left(\int f_x \, d\nu \right) d\mu.$$

22 Example An application of Fubini's theorem to counting measure on $\mathbb{N} \times \mathbb{N}$ implies that if $\{a_{mn} \in \mathbb{C} \mid m, n \in \mathbb{N}\}$ is a doubly-indexed sequence of complex numbers such that

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} |a_{mn}| \right) < \infty$$

then

$$\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{mn} \right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_{mn} \right).$$

Finally, we state a version of Fubini's theorem for functions on $\mathbb{R}^d$.

23 Theorem A Lebesgue measurable function $f : \mathbb{R}^{m+n} \to \mathbb{C}$ is integrable, meaning that

$$\int_{\mathbb{R}^{m+n}} |f(x,y)|\, dxdy < \infty,$$

if and only if either one of the iterated integrals

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x,y)|\, dx \right) dy, \qquad \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x,y)|\, dy \right) dx$$

is finite. In that case,

$$\int_{\mathbb{R}^{m+n}} f(x,y)\, dxdy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)\, dx \right) dy = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)\, dy \right) dx,$$

where all of the integrals are well-defined and finite a.e..

L^p-spaces

Suppose that $(X, \mathcal{A}, \mu)$ is a measure space. If $f : X \to \mathbb{C}$ is a measurable function and $\int |f|\, d\mu = 0$ then it does not follow that $f = 0$, only that the set on which f is non-zero has measure zero. We will henceforth regard two measurable functions that differ from each other on a set of measure zero as the 'same' function. Thus, we identify a function with the equivalence class to which it belongs under the equivalence relation of a.e.-equality. For example, the characteristic function $\chi_{\mathbb{Q}}$ of the rationals on $\mathbb{R}$ is equivalent to 0. With this convention, $\int |f|\, d\mu = 0$ implies that $f = 0$.

For $1 \leq p < \infty$, we define the space $L^p(X)$ to consist of all (equivalence classes of) measurable functions $f : X \to \mathbb{C}$ such that

$$\int |f|^p\, d\mu < \infty.$$

For $f \in L^p(X)$, we define

$$\|f\|_p = \left(\int |f|^p\, d\mu \right)^{1/p}.$$

One can also define the space $L^\infty(X)$ of essentially bounded measurable functions, with norm given by the essential supremum. (Here 'essential' means 'up to sets of measure zero', but we omit the details.)

24 Theorem If $1 \leq p < \infty$, then $L^p(X)$ equipped with the norm $\|\cdot\|_p$ is a Banach space.

The theorem includes the statement that the L^p-norm satisfies the triangle inequality,

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

This is called Minkowski's inequality and it does not hold for $p < 1$, which explains the restriction on p. It also includes the statement that L^p is complete, meaning that if $(f_n)_{n=1}^{\infty}$ is any Cauchy sequence in L^p, then there exists $f \in L^p$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$. The sequence need not converge pointwise a.e., but there is a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $f_{n_k} \to f$ pointwise a.e. as $k \to \infty$. Any function in L^p is the pointwise a.e. and L^p-limit of a sequence of simple functions.

Finally, we give a theorem which states that L^p-functions on $\mathbb{R}^d$ can be approximated by continuous functions. We consider functions $f : \Omega \to \mathbb{C}$ where $\Omega \subset \mathbb{R}^d$ is an arbitrary nonempty open set. We denote by $C_c(\Omega)$ the set of continuous functions $f : \Omega \to \mathbb{C}$ with *compact support* in Ω, meaning that the closure in Ω of the set $\{x \in \Omega \mid f(x) \neq 0\}$ is a compact set. A function $f \in C_c(\Omega)$ is bounded and nonzero on a bounded set, so $f \in L^p(\Omega)$.

25 Theorem $C_c(\Omega)$ is a dense subset of $L^p(\Omega)$.

More explicitly, this theorem states that if $f \in L^p(\Omega)$ then, given any $\epsilon > 0$, there exists $g \in C_c(\Omega)$ such that

$$\|f - g\|_p < \epsilon.$$

It follows that $L^p(\Omega)$ is the completion of $C_c(\Omega)$ with respect to the L^p-norm.

References

A readable, insightful introduction to Lebesgue measure is given in:

E. M. Stein and R. Shakarchi, *Real Analysis*, Princeton, 2005.

For a leisurely, clear, and well-motivated introduction to Lebesgue measure and integration on $\mathbb{R}^n$ see:

F. Jones, *Lebesgue Integration on Euclidean Space*, Jones & Bartlett 1993.

A detailed, individual discussion of measure theory on $\mathbb{R}^n$ is given in:

E. H. Lieb and M. Loss, *Analysis*, AMS 1997.

An excellent, but more condensed, standard text is:

G. B. Folland, *Real Analysis*, Wiley, 1984.

Another good, recent text is:

M. E. Taylor, *Measure Theory and Integration*, AMS, 2006.