Midterm: Math 201B
Winter, 2011

1. Say if the following Fourier series represent functions or distributions in $C^\infty(\mathbb{T})$, $C(\mathbb{T})$, $L^2(\mathbb{T})$, or $\mathcal{D}'(\mathbb{T})$:

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sqrt{1+n^4}} e^{inx};$$

$$g(x) \sim \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{1+n^2}} e^{in^2x};$$

$$h(x) \sim \sum_{n=-\infty}^{\infty} e^{-n^4} e^{inx};$$

$$k(x) \sim \sum_{n=-\infty}^{\infty} e^{n^4} e^{inx}.$$

You may use standard theorems proved in class to justify your answers.

2. Define an operator $K : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$Kf(x) = \int_0^x \left[f(y) - \tilde{f} \right] dy, \qquad \tilde{f} = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx.$$

(a) Show that K is a bounded linear operator on $L^2(\mathbb{T})$.

(b) What is the kernel of K?

(c) What is the range of K? Can you characterize it items of Sobolev spaces? Is the range of K closed?

3. Let $\{x_n\}$, $\{y_n\}$ be sequences in a Hilbert space $\mathcal{H}$.

(a) If $x_n \to x$ (strongly) and $y_n \rightharpoonup y$ (weakly) as $n \to \infty$, prove that

$$\langle x_n, y_n \rangle \to \langle x, y \rangle \qquad \text{as } n \to \infty.$$

(b) Prove or give a counter-example: if $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ then

$$\langle x_n, y_n \rangle \to \langle x, y \rangle \qquad \text{as } n \to \infty.$$

4. (a) If $f \in L^1(\mathbb{T})$, show that

$$\hat{f}(n) = \frac{1}{4\pi} \int_{\mathbb{T}} [f(x) - f(x + \pi/n)]\, e^{-inx}\, dx.$$

(b) Suppose that $f \in C(\mathbb{T})$ is Hölder continuous with exponent α, where $0 < \alpha \leq 1$, meaning that there is a constant $M > 0$ such that

$$|f(x+h) - f(x)| \leq M|h|^\alpha \qquad \text{for all } x, h \in \mathbb{T}.$$

Show that there is a constant $C > 0$ such that

$$\left|\hat{f}(n)\right| \leq \frac{C}{|n|^\alpha} \qquad \text{for all nonzero integers } n.$$

For extra credit, if you have time.
(c) If $0 < \alpha < 1$, show that the function

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} e^{i2^k x}$$

is Hölder continuous with exponent α and that $\hat{f}(n) = 1/n^\alpha$ for $n = 2^k$ (so the above result is optimal).

HINT. You can assume the inequality

$$\left|1 - e^{i\theta}\right| \leq |\theta| \qquad \text{for } \theta \in \mathbb{R}.$$