Practice Midterm problems: Math 201B

1. What can you say about the differentiability of the functions with the following Fourier series:

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{1}{(1+n^4)^{1/7}} e^{inx};$$

$$g(x) \sim \sum_{n=-\infty}^{\infty} n e^{-|n|} e^{inx};$$

$$h(x) \sim \sum_{n=1}^{\infty} \frac{1}{3^n} e^{i2^n x}.$$

e.g. How many continuously derivatives can you say exist?

2. Given a function, or sequence, $\hat{\mu} : \mathbb{Z} \to \mathbb{C}$, define an operator M on periodic functions by

$$M\left(\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}\right) = \sum_{n\in\mathbb{Z}} \hat{\mu}(n)\hat{f}(n)e^{inx}.$$

Show that $M : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a bounded linear operator if and only if $\hat{\mu} \in \ell^\infty(\mathbb{Z})$ is bounded. For what $\hat{\mu}$ is M unitary?

3. Define $K : L^2(0,1) \to L^2(0,1)$ by

$$Kf(x) = \int_0^1 e^{x-y} f(y)\, dy.$$

Show that K is bounded. What is K^*? What is the range of K? What is the kernel of K? Does K have closed range?

4. Define $f_n, g_n \in L^2(\mathbb{T})$ by

$$f_n(x) = \cos(nx), \qquad g_n(x) = n\cos(nx).$$

Do the sequences $\{f_n\}$, $\{g_n\}$ converge strongly in $L^2(\mathbb{T})$? How about weakly in $L^2(\mathbb{T})$? How about in the sense of distributions in $\mathcal{D}'(\mathbb{T})$?

5. Suppose that $T \in \mathcal{D}'(\mathbb{T})$ is a non-negative periodic distribution, meaning that $\langle T, \phi\rangle \geq 0$ whenever $\phi \geq 0$, where $\phi \in C^\infty(\mathbb{T})$ is real-valued. Show that there exists a constant C such that

$$|\langle T, \phi\rangle| \leq C\|\phi\|_\infty \qquad \text{for all } \phi \in C^\infty(\mathbb{T}).$$