Problem Set 1
Math 201B, Winter 2011
Due: Friday, Jan 7

You can look up and use any standard results from integration theory *e.g.* Fubini's theorem, Hölder's inequality, and the density of $C(\mathbb{T})$ in $L^p(\mathbb{T})$.

1. If $1 \le p < q < \infty$, show that $L^p(\mathbb{T}) \supset L^q(\mathbb{T})$. Give an example of a function in $L^p(\mathbb{T}) \setminus L^q(\mathbb{T})$.

2. (a) If $f, g \in L^1(\mathbb{T})$, show that $f * g \in L^1(\mathbb{T})$ and

$$\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}.$$

(b) If $f, g \in L^2(\mathbb{T})$ show that

$$\|f * g\|_\infty \le \|f\|_{L^2} \|g\|_{L^2}$$

and deduce that $f * g \in C(\mathbb{T})$.

3. (a) For $f \in L^p(\mathbb{T})$ and $h \in \mathbb{R}$, let $f_h(x) = f(x+h)$ denote the translation of f by h. If $1 \le p < \infty$, show that $f_h \to f$ in $L^p(\mathbb{T})$ as $h \to 0$. Hint. Approximate f by a continuous function.
(b) Give an example to show that this result is not true when $p = \infty$.

4. (a) Compute the Fourier series expansion of

$$f(x) = |x| \qquad \text{for } |x| \le \pi.$$

(b) Use Parseval's theorem to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$