Remarks on Problem Set 1
Math 201B: Winter, 2011

1. If $1 \leq p < q < \infty$, show that $L^p(\mathbb{T}) \supset L^q(\mathbb{T})$.

Remarks.

- This result depends crucially on the fact that $\mathbb{T}$ has finite measure (equal to 2π). For example, there are no inclusions between $L^p(\mathbb{R})$ and $L^q(\mathbb{R})$ when $p \neq q$. Roughly speaking, this is because taking powers of a function makes its local singularities worse but improves its decay at infinity.

- We do, however, have a simple interpolation result: if
$$1 \leq q < p < r \leq \infty,$$
then
$$L^p(\mathbb{R}) \supset L^q(\mathbb{R}) \cap L^r(\mathbb{R})$$
meaning that if $f \in L^q$ and $f \in L^r$, then $f \in L^p$.

2. (a) If $f, g \in L^1(\mathbb{T})$, show that $f * g \in L^1(\mathbb{T})$ and
$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}.$$
(b) If $f, g \in L^2(\mathbb{T})$ show that
$$\|f * g\|_\infty \leq \|f\|_{L^2} \|g\|_{L^2}$$
and deduce that $f * g \in C(\mathbb{T})$.

Remarks.

- A Banach space B with a bilinear, associative product
$$\cdot : B \times B \to B$$
is called a Banach algebra if
$$\|f \cdot g\| \leq \|f\| \, \|g\|.$$
According to (a), the space $L^1(\mathbb{T})$ with the convolution product
$$* : L^1(\mathbb{T}) \times L^1(\mathbb{T}) \to L^1(\mathbb{T})$$
is a Banach algebra. (The convolution product is associative.) In fact, since $f * g = g * f$, it is a commutative Banach algebra.

- More generally, if $f \in L^1$ and $g \in L^p$, then $f * g \in L^p$ and
$$\|f * g\|_{L^p} \leq \|f\|_{L^1} \|g\|_{L^p}.$$

- All these inequalities are special (but important) cases of Young's inequality: If $1 \leq p, q, r \leq \infty$,
$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$
and $f \in L^p$, $g \in L^q$, then $f * g \in L^r$ and
$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

3. (a) For $f \in L^p(\mathbb{T})$ and $h \in \mathbb{T}$, let
$$f_h(x) = f(x + h)$$
denote the translation of f by h. If $1 \leq p < \infty$, show that $f_h \to f$ in L^p as $h \to 0$. Hint. Approximate f by a continuous function.
(b) Give an example to show that this result is not true when $p = \infty$.

Remarks.

- Note that this L^p-continuity does not mean that L^p-functions are continuous!

- A Banach space $B \subset L^1(\mathbb{T})$ with norm $\|\cdot\|_B$ is said to be a homogeneous space if $f_h \in B$ for every $f \in B$ and $h \in \mathbb{T}$ and $f_h \to f$ in B as $h \to 0$. Thus, $L^p(\mathbb{T})$ is a homogenous space for $1 \leq p < \infty$, as is $C(\mathbb{T})$ equipped with the sup-norm, but $L^\infty(\mathbb{T})$ is not a homogeneous space.

- The torus $\mathbb{T}$ is a commutative topological group with continuous group operation $+ : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ where $(x, y) \mapsto x + y$. The exponential functions $e^{inx} : \mathbb{T} \to \mathbb{C}$ are the characters of $\mathbb{T}$ *i.e.* continuous homomorphisms from $\mathbb{T}$ into the multiplicative unit circle in $\mathbb{C}$
$$e^{in(x+y)} = e^{inx} e^{iny}.$$
They are also eigenfunctions of the translation operators $\tau_h : f \mapsto f_h$. The characters of $\mathbb{T}$ form the dual group $\hat{\mathbb{T}}$, which is isomorphic to $\mathbb{Z}$. Many aspects of harmonic analysis on $\mathbb{T}$ can be generalized to the harmonic analysis of functions on a locally compact topological group G in which the Fourier transform of a complex-valued function on G is a function on the dual group $\hat{G}$.

4. (a) Compute the Fourier series expansion of

$$f(x) = |x| \qquad \text{for } |x| \leq \pi.$$

(b) Use Parseval's theorem to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Remarks.

- It is, perhaps, surprising that the completeness of the Fourier basis implies such quantitative results.
- The value of this sum is equal to $\zeta(4)$ where ζ is the Riemann zeta function. This function is defined for $\Re z > 1$ by the convergent series

  $$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

  It may be extended by analytic continuation to a meromorphic function

  $$\zeta : \mathbb{C} \setminus \{1\} \to \mathbb{C}$$

  which has a simple pole at $z = 1$ and is holomorphic (a differentiable function of the complex variable z) in $\mathbb{C} \setminus \{1\}$.
- The values $\zeta(2n)$ of the zeta function at even integers can be computed by the use of Fourier series (or in many other ways) and may be expressed in terms of the Bernoulli numbers. There is no known explicit expression for $\zeta(n)$ when n is an odd integer *e.g.* Apéry's number $\zeta(3)$.
- The Riemann zeta function is equal to zero at even negative integers $z = -2n$ where $n \in \mathbb{N}$. As an exercise in complex analysis, you might try to prove that the only other zeros of the zeta function lie on the line $\Re z = 1/2$.