Remarks on Problem Set 2:
Math 201B: Winter, 2011

1. If $1 \le p < \infty$, show that the trigonometric polynomials are dense in $L^p(\mathbb{T})$.

Remarks.

- This result follows immediately from the density of $C(\mathbb{T})$ in $L^p(\mathbb{T})$ and the Weierstrass approximation theorem that continuous functions can be approximated continuously by trigonometric polynomials. A generalization of this last result is the following complex form of the Stone-Weierstrass theorem.

- Suppose that X is a compact Hausdorff topological space and $\mathcal{A} \subset C(X)$ is a C^*-algebra of continuous complex-valued function on X (*i.e.* $\mathcal{A}$ is a complex linear subspace of $C(X)$ such that $fg \in \mathcal{A}$ if $f, g \in \mathcal{A}$ and$\bar{f} \in \mathcal{A}$ if $f \in \mathcal{A}$) such that: (a) $\mathcal{A}$ contains the constant functions; (b) $\mathcal{A}$ separates points (*i.e.* if $x, y \in X$ and $x \ne y$ then there exists $f \in \mathcal{A}$ such that $f(x) \ne f(y)$). Then $\mathcal{A}$ is dense in $C(X)$ with respect to the uniform norm.

2. For fixed $z \in \mathbb{C}$, let $J_n(z)$ denote the nth Fourier coefficient of the function $e^{iz\sin x}$, meaning that

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin x} e^{-inx}\, dx \qquad \text{for } n \in \mathbb{Z}.$$

Remarks.

- The functions $J_n(z)$ are the Bessel functions of integer order n. The Fourier expression can also be written in the form of a generating function as
$$\exp\left[\frac{1}{2}z\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)t^n.$$

- Bessel functions are one of the most important special functions. They arise, for example, in finding separable solutions of the wave and Schrödinger equations in cylindrical and spherical coordinates. The classic book by G. N. Watson, *A Treatise on the Theory of Bessel Functions* (1922), was partly the result of Watson's work on the analysis of the diffraction of radio waves around the spherical earth.

3. A family of (not necessarily positive) functions $\{\phi_n \in L^1(\mathbb{T}) : n \in \mathbb{N}\}$ is an approximate identity if:

$$\int \phi_n \, dx = 1 \qquad \text{for every } n \in \mathbb{N};$$

$$\int |\phi_n| \, dx \leq M \qquad \text{for some constant } M \text{ and all } n \in \mathbb{N};$$

$$\lim_{n\to\infty} \int_{\delta<|x|<\pi} |\phi_n| \, dx = 0 \qquad \text{for every } \delta > 0.$$

If $f \in L^1(\mathbb{T})$, show that $\phi_n * f \to f$ in $L^1(\mathbb{T})$ as $n \to \infty$.

Remarks.

- The analogous conclusion holds for L^p with $1 \leq p < \infty$: If $f \in L^p(\mathbb{T})$, then $\phi_n * f \to f$ in $L^p(\mathbb{T})$ as $n \to \infty$.
- It is not, in general, true that $\phi_n * f \to f$ pointwise a.e., unless we impose some additional conditions on $\{\phi_n\}$. The proofs of pointwise a.e. convergence under appropriate hypotheses are more subtle than the ones for norm convergence and typically involve the use of maximal functions.

4. Let $\{a_n : n \geq 0\}$ be a sequence of non-negative real numbers such that $a_n \to 0$ as $n \to \infty$ and

$$a_{n+1} - 2a_n + a_{n-1} \geq 0. \tag{1}$$

For $N \geq 0$, let $K_N \geq 0$ denote the Fejér kernel. Show that the series

$$f(x) = \sum_{n=1}^{\infty} n\,(a_{n+1} - 2a_n + a_{n-1})\,K_{n-1}(x)$$

converges in $L^1(\mathbb{T})$ to a non-negative function $f \in L^1(\mathbb{T})$ whose Fourier coefficients are $a_{|n|}$. Show that there is no function $f \in L^1(\mathbb{T})$ such that

$$f(x) \sim \sum_{|n|\geq 2} \frac{i \operatorname{sgn} n}{\log |n|} e^{inx}.$$

Remarks.

- This example is given in Zygmund's and Katznelson's books. It shows that the Fourier transform $\mathcal{F} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ is not onto since there exist sequences converging to zero that are not the Fourier coefficients of L^1-functions.

- It is difficult to characterize explicitly the sequences in c_0 that are the Fourier coefficients of L^1-functions. For example, given any positive sequence $\{c_n\}$ such that $c_n \to 0$ as $n \to \infty$, however slowly, there exists a positive sequence $\{a_n\}$ satisfying the convexity condition (1) such that $a_n \geq c_n$ for all $n \in \mathbb{N}$, and therefore an L^1-function whose Fourier coefficients are $a_{|n|}$. Thus, although the Fourier coefficients of an L^1 function approach zero as $n \to \infty$, they may do so arbitrarily slowly.