**Problem Set 3: Math 201B**

Due: Friday, January 21

**1.** Suppose that $\sum_{n=0}^{\infty} c_n$ is a series of complex numbers with partial sums

$$s_n = \sum_{k=0}^{n} c_k.$$

The series is Borel summable with Borel sum $s$ if the following limit exists:

$$s = \lim_{x \to +\infty} e^{-x} \left( \sum_{n=0}^{\infty} \frac{s_n x^n}{n!} \right).$$

(a) If the series $\sum_{n=0}^{\infty} c_n = s$ is convergent, show that it is Borel summable with Borel sum equal to $s$ (meaning that Borel summation is regular).

(b) For what complex numbers $a \in \mathbb{C}$ is the geometric series

$$\sum_{n=0}^{\infty} a^n$$

Borel summable? What is its Borel sum? For what $a \in \mathbb{C}$ is this series Cesàro summable? Abel summable?

(c) Do you get anything useful from the Borel summation of a Fourier series?

**2.** Let $A(\mathbb{T})$ denote the space of integrable functions whose Fourier coefficients are absolutely convergent. That is, $f \in A(\mathbb{T})$ if

$$\sum_{n \in \mathbb{Z}} \left| \hat{f}(n) \right| < \infty.$$

(a) If $f \in A(\mathbb{T})$, show that $f \in C(\mathbb{T})$. Also show that $f \in A(\mathbb{T})$ if and only if $f = g * h$ for some functions $g, h \in L^2(\mathbb{T})$.

(b) If $f, g \in A(\mathbb{T})$, show that $fg \in A(\mathbb{T})$ and express $\widehat{fg}$ in terms of $\hat{f}$, $\hat{g}$.

**Optional question!**

(c) Give an example of a function $f \in C(\mathbb{T})$ such that $f \notin A(\mathbb{T})$.

**3.** Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc in the complex plane. The Hardy space $H^2(D)$ is the space of functions with a power series expansion

$$F(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1}$$

such that

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty. \tag{2}$$

This is a Hilbert space with inner product

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle = \sum_{n=0}^{\infty} \bar{a}_n b_n.$$

(a) If (2) holds, show that the power series (1) converges in $D$ to a holomorphic (analytic) function $F : D \to \mathbb{C}$. (You can use standard definitions and facts from complex analysis.)

(b) Is $1/(1-z) \in H^2(D)$? If $\theta_0 \in \mathbb{T}$, give an example of a function $F \in H^2(D)$ which does not extend to a function that is analytic at $z = e^{i\theta_0}$.

(c) If $F \in H^2(D)$, show that

$$\|F\|_{H^2}^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| F\left(re^{i\theta}\right) \right|^2 \, d\theta < \infty.$$

Show conversely that if $F : D \to \mathbb{C}$ is a holomorphic function such that

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| F\left(re^{i\theta}\right) \right|^2 \, d\theta < \infty$$

then $F \in H^2(D)$.

(d) Let

$$\widetilde{H}^2(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) : \hat{f}(n) = 0 \text{ for } n < 0 \right\}.$$

If $F \in H^2(D)$ is given by (1) and $0 < r < 1$, define $f_r \in L^2(\mathbb{T})$ by

$$f_r(\theta) = F(re^{i\theta}).$$

Show that $f_r \to f$ as $r \to 1^-$ in $L^2(\mathbb{T})$ where

$$f(\theta) = \sum_{n=0}^{\infty} c_n e^{in\theta} \in \widetilde{H}^2(\mathbb{T}).$$

Conversely, if $f \in \widetilde{H}^2(\mathbb{T})$, define $F : D \to \mathbb{C}$ by

$$F(re^{i\theta}) = (P_r * f)(\theta)$$

where $P_r$ is the Poisson kernel. Show that $F \in H^2(\mathbb{T})$.