Remarks on Problem Set 3
Math 201B: Winter 2011

1. Suppose that $\sum_{n=0}^{\infty} c_n$ is a series of complex numbers with partial sums

$$s_n = \sum_{k=0}^{n} c_k.$$

The series is Borel summable with Borel sum s if the following limit exists:

$$s = \lim_{x \to +\infty} e^{-x} \left(\sum_{n=0}^{\infty} \frac{s_n x^n}{n!} \right).$$

(b) For what complex numbers $a \in \mathbb{C}$ is the geometric series

$$\sum_{n=0}^{\infty} a^n$$

Borel summable? What is its Borel sum? For what $a \in \mathbb{C}$ is this series Cesàro summable? Abel summable?

(c) Do you get anything useful from the Borel summation of a Fourier series?

Remarks.

- As (b) illustrates, Borel summation can give the analytic continuation of a power series outside its radius of convergence, in this case from the disc $|a| < 1$ to the half-plane $\Re a < 1$. Roughly speaking, Borel summation is a more powerful — but cruder — method of summing divergent series than Abel or Cesàro summation, which can only sum a power series on the boundary $|a| = 1$ of its disc of convergence.

- Borel summation has been used to re-sum divergent perturbation series that arise in quantum field theory and from various PDEs.

- (c) The short answer is no. A longer answer is in the paper by Moore.

2. Let $A(\mathbb{T})$ denote the space of integrable functions whose Fourier coefficients are absolutely convergent. That is, $f \in A(\mathbb{T})$ if

$$\sum_{n \in \mathbb{Z}} \left| \hat{f}(n) \right| < \infty.$$

(a) If $f \in A(\mathbb{T})$, show that $f \in C(\mathbb{T})$. Also show that $f \in A(\mathbb{T})$ if and only if $f = g * h$ for some functions $g, h \in L^2(\mathbb{T})$.

(b) If $f, g \in A(\mathbb{T})$, show that $fg \in A(\mathbb{T})$ and express $\widehat{fg}$ in terms of $\hat{f}$, $\hat{g}$.

Optional question!

(c) Give an example of a function $f \in C(\mathbb{T})$ such that $f \notin A(\mathbb{T})$.

Remarks.

- (a) The space $A(\mathbb{T})$ of functions with absolutely convergent Fourier series is not so easy to characterize explicitly.

- As (b) shows, $A(\mathbb{T})$ is an algebra with respect to the pointwise product. This algebra maps under the Fourier transform to $\ell^1(\mathbb{Z})$ with the discrete convolution product: if $\hat{f} : \mathbb{Z} \to \mathbb{C}$, $\hat{g} : \mathbb{Z} \to \mathbb{C}$ belong to $\ell^1(\mathbb{Z})$ then $\hat{f} * \hat{g} : \mathbb{Z} \to \mathbb{C}$ in $\ell^1(\mathbb{Z})$ is defined by

$$\left(\hat{f} * \hat{g}\right)(k) = \sum_{n \in \mathbb{Z}} \hat{f}(k-n)\hat{g}(n).$$

  This is dual to the fact that $L^1(\mathbb{T})$ is an algebra with respect to the convolution product, and the Fourier transform maps $L^1(\mathbb{T})$ — analogous to $\ell^1(\mathbb{Z})$ — to a sequence subspace of $c_0(\mathbb{Z})$ — analogous to the subspace $A(\mathbb{T})$ of $C(\mathbb{T})$, with the pointwise product.

- (c) Continuous functions with non-absolutely convergent Fourier coefficients are constructed in Zygmund's book on Trigonometric Series *c.f.* Theorem 4.2 in Chapter V. One example is

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{in \log n}}{n} e^{inx}.$$

  Even though this Fourier series is not absolutely convergent, it converges uniformly to f, and f is not only continuous but Hölder continuous with exponent $1/2$, meaning that

$$|f(x) - f(y)| \leq C|x-y|^{1/2}.$$

  There are also continuous function whose Fourier series diverge at a point, or on a set of measure zero, as well as continuous functions whose Fourier series converge pointwise everywhere but do not converge uniformly *c.f.* Section 1, Chapter VIII of Zygmund.

3. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc in the complex plane. The Hardy space $H^2(D)$ is the space of functions with a power series expansion

$$F(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1}$$

such that

$$\sum_{n=0}^{\infty} |c_n|^2 < \infty. \tag{2}$$

If $F \in H^2(D)$, show that

$$\|F\|_{H^2}^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| F\left(re^{i\theta}\right) \right|^2 \, d\theta < \infty.$$

Show conversely that if $F : D \to \mathbb{C}$ is a holomorphic function such that

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| F\left(re^{i\theta}\right) \right|^2 \, d\theta < \infty$$

then $F \in H^2(D)$.

Remarks.

- More generally, if $0 < p < \infty$, the Hardy space $H^p(D)$ is defined to be the set of analytic functions $F : D \to \mathbb{C}$ that satisfy the following growth condition at the boundary of D:

$$\sup_{0<r<1} \int_0^{2\pi} \left| F\left(re^{i\theta}\right) \right|^p \, d\theta < \infty.$$

  The Hardy space $H^\infty(D)$ consists of the bounded analytic functions on D. The only one of these Hardy spaces that is a Hilbert space is $H^2(D)$.

- For $1 < p \leq \infty$, the Hardy spaces are essentially equivalent to $L^p(\mathbb{T})$, but this is no longer true for $0 < p \leq 1$. A great deal of effort in harmonic analysis has been made to understand the structure of functions in these Hardy spaces, originally by the use of complex-variable methods and subsequently by the use of real-variable methods (which extend to functions of more than one variable).