Problem Set 4: Math 201B

Due: Friday, January 28

1. Let $D \subset \mathbb{R}^2$ be the unit disc and $f \in C(\partial D)$ a continuous function defined on the unit circle ∂D. Suppose that $u : \overline{D} \to \mathbb{R}$ is a function $u \in C^2(D) \cap C(\overline{D})$ such that

$$\begin{aligned} \Delta u = 0 \qquad & \text{in } D, \\ u = f \qquad & \text{on } \partial D. \end{aligned} \tag{1}$$

(a) Show that

$$\max_{\overline{D}} u = \max_{\partial D} f.$$

HINT. Let $u^\epsilon(x, y) = u(x, y) + \epsilon(x^2 + y^2)$ and show that u^ϵ cannot have an interior maximum for any $\epsilon > 0$.

(b) Deduce that a solution of (1) is unique and is therefore given by

$$u(r, \theta) = (P_r * f)(\theta)$$

in $0 \le r < 1$ where P_r is the Poisson kernel.

2. Define $f \in L^2(\mathbb{T})$ by

$$f(x) = |x| \qquad \text{for } |x| < \pi.$$

Show that $f \in H^1(\mathbb{T})$ and compute its weak derivative $f' \in L^2(\mathbb{T})$. Is $f' \in H^1(\mathbb{T})$? For what values of $s > 0$ is it true that $f \in H^s(\mathbb{T})$?

3. Suppose that $f : [0, L] \to \mathbb{R}$ is a smooth function *e.g.* $f \in C^1([0, L])$ such that $f(0) = f(L) = 0$. Prove that

$$\int_0^L [f(x)]^2 \, dx \le \left(\frac{L}{\pi}\right)^2 \int_0^L \left[f'(x)\right]^2 \, dx.$$

Show that the constant in this inequality is sharp. Why do you need to assume that $f(0) = f(L) = 0$? Show that you cannot estimate the L^2-norm of a smooth, square-integrable function $f : [0, \infty) \to \mathbb{R}$ such that $f(0) = 0$ in terms of the L^2 norm of its derivative.

4. Suppose that $u(x,t)$ is a solution of the following initial value problem for the heat equation

$$\begin{aligned} u_t &= u_{xx} && x \in \mathbb{T},\ t > 0 \\ u(x,0) &= f(x) && x \in \mathbb{T} \end{aligned}$$

where $f \in C(\mathbb{T})$ and

$$u \in C^2\left(\mathbb{T} \times (0,\infty)\right) \cap C\left(\mathbb{T} \times [0,\infty)\right).$$

(a) Show that

$$u(x,t) = \left(\theta_t * f\right)(x) \qquad \text{for } t > 0$$

where

$$\theta_t(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-n^2 t} e^{inx}.$$

(b) Show that $u \in C^\infty\left(\mathbb{T} \times (0,\infty)\right)$.