Remarks on Problem Set 4
Math 201B: Winter 2001

1. Let $D \subset \mathbb{R}^2$ be the unit disc and $f \in C(\partial D)$ a continuous function defined on the unit circle ∂D. Suppose that $u : \overline{D} \to \mathbb{R}$ is a function $u \in C^2(D) \cap C(\overline{D})$ such that

$$\begin{aligned} \Delta u = 0 \qquad & \text{in } D, \\ u = f \qquad & \text{on } \partial D. \end{aligned} \tag{1}$$

Show that

$$\max_{\overline{D}} u = \max_{\partial D} f.$$

Hint. Let $u^\epsilon(x, y) = u(x, y) + \epsilon(x^2 + y^2)$ and show that u^ϵ cannot have an interior maximum for any $\epsilon > 0$.

Remarks.

- This result is called the weak maximum principle for harmonic functions. Note that we have to perturb u slightly to u^ϵ to take care of the possibility of a degenerate maximum where $u_{xx} = u_{yy} = 0$. The strong maximum principle states that a harmonic function which attains a local maximum at an interior point of a connected open set D must be constant in D. Similar maximum principles apply to solutions of other scalar elliptic PDEs.

- A physical interpretation of this result is that if a disc in in thermal equilibrium then, in the absence of heat sources, the temperature is greatest and least at some points on the boundary and lies between these extreme values everywhere in the interior of the disc.

2. Define $f \in L^2(\mathbb{T})$ by

$$f(x) = |x| \qquad \text{for } |x| < \pi.$$

Show that $f \in H^1(\mathbb{T})$ and compute its weak derivative $f' \in L^2(\mathbb{T})$. Is $f' \in H^1(\mathbb{T})$? For what values of $s > 0$ is it true that $f \in H^s(\mathbb{T})$?

Remarks.

- As this example illustrates, a function like $f'(x) = \operatorname{sgn} x$ may be differentiable pointwise *a.e.* but not weakly differentiable; the weak derivative is the 'right' way to define the derivative of a function which isn't smooth. The function f' does, however, have a distributional derivative, $f'' = 2\delta$, but this is not a weak derivative since the periodic δ-function is not a regular distribution.

3. Suppose that $f : [0, L] \to \mathbb{R}$ is a smooth function *e.g.* $f \in C^1([0, L])$ such that $f(0) = f(L) = 0$. Prove that

$$\int_0^L [f(x)]^2 \, dx \le \left(\frac{L}{\pi}\right)^2 \int_0^L [f'(x)]^2 \, dx.$$

Show that the constant in this inequality is sharp. Why do you need to assume that $f(0) = f(L) = 0$? Show that you cannot estimate the L^2-norm of a smooth, square-integrable function $f : [0, \infty) \to \mathbb{R}$ such that $f(0) = 0$ in terms of the L^2 norm of its derivative.

Remarks.

- This inequality is called Wirtinger's inequality. It is the simplest example of a Poincaré inequality that estimates the L^2 norm of a function in terms of the L^2 norm of its derivative under suitable assumptions that eliminate the constant functions and imposes some boundedness on the domain of the function.

- For example, if $\Omega \subset \mathbb{R}^n$ is a bounded open set and $u : \Omega \to \mathbb{R}$ is a smooth function with compact support in Ω, then

$$C \int_\Omega u^2 \, dx \le \int_\Omega |\nabla u| \, dx$$

where $C > 0$ is a constant independent of u. The best constant C is, in fact, the smallest eigenvalue of the Dirichlet Laplacian $-\Delta$ on Ω.

- In the given problem, the eigenvalues λ_n and eigenfunctions f_n of $-d^2/dx^2$ with Dirichlet boundary conditions

$$-f_n'' = \lambda_n f_n, \qquad f_n(0) = f_n(L) = 0$$

are given by

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad f_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

for $n \in \mathbb{N}$, and $C = \lambda_1 = \pi^2/L^2$.

4. Suppose that $u(x,t)$ is a solution of the following initial value problem for the heat equation

$$\begin{aligned} u_t &= u_{xx} \qquad && x \in \mathbb{T},\ t > 0 \\ u(x,0) &= f(x) && x \in \mathbb{T} \end{aligned}$$

where $f \in C(\mathbb{T})$ and

$$u \in C^2\left(\mathbb{T} \times (0,\infty)\right) \cap C\left(\mathbb{T} \times [0,\infty)\right).$$

Show that

$$u(x,t) = \left(\theta_t * f\right)(x) \qquad \text{for } t > 0$$

Remarks.

- The function

$$\theta_t(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{inx}$$

is the Green's function of the heat equation with periodic boundary conditions. It is a smooth solution of the heat equation in $t > 0$ and, in the sense of periodic distributions, we have

$$\theta_t(x) \rightharpoonup \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{inx} = \delta(x) \qquad \text{in } \mathcal{D}'(\mathbb{T}) \text{ as } t \to 0^+.$$

- Writing $t = -i\pi\tau$ and $x = 2\pi z$, we can extend $2\pi\theta_t(x)$ to a function $\vartheta(z;\tau)$ of the complex variables (z,τ) defined by

$$\vartheta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

This series converge uniformly on compact sets in $\mathbb{C} \times \{\Im\tau > 0\}$ to a holomorphic function, called a theta function.

- Note that

$$\vartheta(z+1;\tau) = \vartheta(z;\tau), \qquad \vartheta(z+\tau;\tau) = e^{-\pi i\tau - 2\pi i z}\vartheta(z;\tau).$$

  Thus, up to some factors, the theta function is almost doubly-periodic in z, with period 1 and quasi-period τ, and one can use it to construct doubly-periodic meromorphic functions of z (elliptic functions). Theta functions have many connections with Riemann surfaces, integrable systems, number theory (the Riemann zeta function), automorphic forms, and other areas of mathematics.