Problem Set 5: Math 201B

Due: Friday, February 4

1. Let $\mathbb{T}^d = \mathbb{T} \times \mathbb{T} \times \cdots \times \mathbb{T}$ denote the d-dimensional Torus.

(a) Show that $\mathcal{B} = \left\{ e^{i\mathbf{n}\cdot\mathbf{x}} : \mathbf{n} \in \mathbb{Z}^d \right\}$ is an orthogonal set in $L^2(\mathbb{T}^d)$ and give an expression for the Fourier coefficients $\hat{f}(\mathbf{n})$ of a function

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \hat{f}(\mathbf{n}) e^{i\mathbf{n}\cdot\mathbf{x}} \in L^2(\mathbb{T}^d).$$

(You can assume that $\mathcal{B}$ is complete — the proof is similar to the one-dimensional case *e.g.* use an approximate identity

$$\Phi_n(\mathbf{x}) = \phi_n(x_1)\phi_n(x_2)\ldots\phi_n(x_d) \qquad n \in \mathbb{N}$$

that is a product of one-dimensional approximate identities $\{\phi_n\}$ consisting of trigonometric polynomials.)

(b) For $s > 0$, let $H^s(\mathbb{T}^d)$ denote the space of functions $f \in L^2(\mathbb{T}^d)$ such that

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \left(1 + |\mathbf{n}|^{2s}\right) \left|\hat{f}(\mathbf{n})\right|^2 < \infty.$$

Prove that if $s > d/2$ and $f \in H^s(\mathbb{T}^d)$, then $f \in C(\mathbb{T}^d)$.

2. (a) Show that any test function $\phi \in C^\infty(\mathbb{T})$ can be written as $\phi = c + \psi'$ where

$$c = \frac{1}{2\pi}\int \phi \, dx, \qquad \psi \in C^\infty(\mathbb{T}).$$

(b) Suppose that $f \in L^1(\mathbb{T})$ is weakly differentiable and its weak derivative $f' = 0$ is zero. Prove that $f = $ constant (up to pointwise a.e. equivalence).

3. Define the principal-value functional $T : \mathcal{D}(\mathbb{T}) \to \mathbb{C}$ by

$$\begin{aligned}\langle T, \phi \rangle &= \text{p.v.} \int_{\mathbb{T}} \cot\left(\frac{x}{2}\right) \phi(x)\, dx \\ &= \lim_{\epsilon \to 0^+} \left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi} \right) \cot\left(\frac{x}{2}\right) \phi(x)\, dx.\end{aligned}$$

(a) Show that $T \in \mathcal{D}'(\mathbb{T})$ is a well-defined periodic distribution.

(b) Compute the Fourier coefficients $\hat{T}(n)$ of T.

4. (a) If $T \in \mathcal{D}'(\mathbb{T})$ is a periodic distribution, show that there exists an integer $k \geq 0$ and a constant C such that

$$|\langle T, \phi \rangle| \leq C \, \|\phi\|_{C^k} \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{T}) \tag{1}$$

where

$$\|\phi\|_{C^k} = \sum_{j=0}^{k} \sup_{x \in \mathbb{T}} \left| \phi^{(j)}(x) \right|$$

denotes the C^k-norm of ϕ.

(b) The order of a distribution T is the minimal integer $k \geq 0$ such that (1) holds. What it the order of: (i) a regular distribution; (ii) the delta-function; (iii) the principal value distribution in the previous question? Give an example of a distribution of order 100.