Remarks on Problem Set 5

Math201B: Winter 2011

1. Let $\mathbb{T}^d = \mathbb{T} \times \mathbb{T} \times \cdots \times \mathbb{T}$ denote the d-dimensional Torus. For $s > 0$, let $H^s(\mathbb{T}^d)$ denote the space of functions $f \in L^2(\mathbb{T}^d)$ such that

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} \left(1 + |\mathbf{n}|^{2s}\right) \left|\hat{f}(\mathbf{n})\right|^2 < \infty.$$

Prove that if $s > d/2$ and $f \in H^s(\mathbb{T}^d)$, then $f \in C(\mathbb{T}^d)$.

Remarks.

- Roughly speaking, this result means that the existence of more that one-half a weak L^2-derivative per space dimension implies that the function is continuous.
- Analogous Sobolev embedding theorems hold for L^p-derivatives, in which case at least $k > d/p$ weak L^p derivatives (more than $1/p$ derivatives per space dimension) are required to imply continuity. When $p \neq 2$, however, a proof using Fourier analysis is not straightforward.

2. Suppose that $f \in L^1(\mathbb{T})$ is weakly differentiable and its weak derivative $f' = 0$ is zero. Prove that $f =$ constant (up to pointwise a.e. equivalence).

Remarks.

- The same result, with essentially the same proof, holds for distributions: If $T \in \mathcal{D}'(\mathbb{T})$ has zero distributional derivative, then T is constant.

3. Define the principal-value functional $T : \mathcal{D}(\mathbb{T}) \to \mathbb{C}$ by

$$\begin{aligned}\langle T, \phi \rangle &= \text{p.v.} \int_{\mathbb{T}} \cot\left(\frac{x}{2}\right) \phi(x)\, dx \\ &= \lim_{\epsilon \to 0^+} \left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi} \right) \cot\left(\frac{x}{2}\right) \phi(x)\, dx.\end{aligned}$$

(a) Show that $T \in \mathcal{D}'(\mathbb{T})$ is a well-defined periodic distribution.
(b) Compute the Fourier coefficients $\hat{T}(n)$ of T.

Remarks.

- The principal value distribution

$$\text{p.v.} \frac{1}{\pi} \cot\left(\frac{x}{2}\right) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} (-i \operatorname{sgn} n)\, e^{inx}$$

is the kernel of the periodic Hilbert transform

$$H : L^2(\mathbb{T}) \to L^2(\mathbb{T})$$

defined by the convolution

$$Hf(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \cot\left(\frac{x-y}{2}\right) f(y)\, dy.$$

The Fourier expression is

$$\widehat{(Hf)}(n) = -i \operatorname{sgn} n \hat{f}(n)$$

i.e. multiplication by $-i \operatorname{sgn} n$. The Hilbert transform is a unitary map on the space of periodic L^2-functions with zero mean, and is a basic example of a singular integral operator.

4. (a) If $T \in \mathcal{D}'(\mathbb{T})$ is a periodic distribution, show that there exists an integer $k \geq 0$ and a constant C such that

$$|\langle T, \phi \rangle| \leq C \, \|\phi\|_{C^k} \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{T}) \tag{1}$$

where

$$\|\phi\|_{C^k} = \sum_{j=0}^{k} \sup_{x \in \mathbb{T}} \left| \phi^{(j)}(x) \right|$$

denotes the C^k-norm of ϕ.

Remarks.

- This result depends on the compactness of $\mathbb{T}$. An example of a distribution $T \in \mathcal{D}'(\mathbb{R})$ that does not have finite order is given by

  $$T = \sum_{n=1}^{\infty} \delta^{(n)}(x - n),$$

  meaning that T is a sum of derivatives of δ-functions whose order increases as their support moves further away from the origin.