Problem Set 6: Math 201B

Due: Friday, February 11

1. Let X be a (real or complex) linear space and $P, Q : X \to X$ projections.

(a) Show that $I - P$ is the projection onto $\ker P$ along $\operatorname{ran} P$.

(b) The projections P, Q are orthogonal, written $P \perp Q$, if $PQ = QP = 0$. Show that $P + Q$ is a projection if and only if $P \perp Q$.

(c) If the projections P, Q commute, show that PQ is the projection onto $\operatorname{ran} P \cap \operatorname{ran} Q$ along $\ker P + \ker Q$.

(d) Give an example (or examples) to show that $P + Q$ need not be a projection if $PQ = 0$ but $QP \neq 0$, and PQ need not be a projection if P,Q do not commute.

2. Let $\mathcal{H} = L^2(\mathbb{R})$. For any Lebesgue measurable set $A \subset \mathbb{R}$, define

$$P_A : \mathcal{H} \to \mathcal{H}$$

by $P_A f = \chi_A f$ where χ_A is the characteristic function of A. (We define $P_\emptyset = 0$.) Show that P_A is an orthogonal projection. What are its range and kernel? Show that P_A, P_B commute. What is $P_A P_B$? When is $P_A \perp P_B$? What is $P_A + P_B$ in that case?

3. Suppose that $\mathcal{H}$ is a separable Hilbert space with ON basis $\{e_n : n \in \mathbb{N}\}$. Let M be the closed linear span of

$$e_1, \quad e_3, \quad e_5, \quad e_7, \quad \ldots$$

and N the closed linear span of

$$e_1 + \frac{1}{2}e_2, \quad e_3 + \frac{1}{2^2}e_4, \quad e_5 + \frac{1}{2^3}e_6, \quad e_7 + \frac{1}{2^3}e_8 \quad \ldots.$$

(a) Show that $M \cap N = \{0\}$. If $X = M \oplus N$, show that

$$\overline{X} = \mathcal{H}, \qquad X \neq \mathcal{H}.$$

(Thus, X is an inner-product space when equipped with the $\mathcal{H}$-inner-product.)

(b) Let $P : X \to X$ be the projection of X onto M along N. Show that P is unbounded.

4. Let $\mathcal{H} = H^1(\mathbb{T})$ denote the Sobolev space of 2π-periodic functions in $L^2(\mathbb{T})$ whose weak derivative belongs to $L^2(\mathbb{T})$ with inner product

$$\langle u, v \rangle_{\mathcal{H}} = \int_{\mathbb{T}} \left(\bar{u} v + \bar{u}' v' \right) \, dx.$$

For $f \in L^2(\mathbb{T})$, define $F : \mathcal{H} \to \mathbb{C}$ by

$$F(v) = \int_{\mathbb{T}} \bar{f} v \, dx.$$

Show that $F \in \mathcal{H}^*$ and find the element $u \in \mathcal{H}$ such that

$$F(v) = \langle u, v \rangle_{\mathcal{H}}.$$

What is $\|F\|_{\mathcal{H}^*}$?