Problem Set 7: Math 201B

Due: Friday, February 18

1. Let $\mathcal{H} = L^2(0,1)$ with the standard inner product

$$\langle f, g \rangle = \int_0^1 \bar{f}(x) g(x)\, dx.$$

Define $M : \mathcal{H} \to \mathcal{H}$ by

$$(Mf)(x) = xf(x)$$

i.e. M is multiplication by x.

(a) Show that M is a bounded self-adjoint linear operator on $\mathcal{H}$ and find $\|M\|$.

(b) What is the kernel of M? What is the range of M? Is M onto? Is $\operatorname{ran} M$ closed?

2. Let $\mathcal{H}$ be a complex Hilbert space.

(a) If $A, B \in \mathcal{B}(\mathcal{H})$ are bounded linear operators on $\mathcal{H}$ such that

$$\langle x, Ax \rangle = \langle x, Bx \rangle \qquad \text{for all } x \in \mathcal{H}$$

show that $A = B$.

(b) Show that an operator $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint if and only if $\langle x, Ax \rangle$ is real for all $x \in \mathcal{H}$.

(c) Do these results remain true if $\mathcal{H}$ is a real Hilbert space?

3. Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded, self-adjoint linear operator such that there is a constant $c > 0$ with

$$c\|x\| \le \|Ax\| \qquad \text{for all } x \in \mathcal{H}.$$

Prove that there is a unique solution x of the equation $Ax = y$ for every $y \in \mathcal{H}$.

4. A Laurent operator (or discrete convolution) is a bounded linear operator A on $\ell^2(\mathbb{Z})$ whose matrix with respect to the standard basis is

$$[A] = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_0 & a_{-1} & a_{-2} & \cdot \\ \cdot & a_1 & a_0 & a_{-1} & \cdot \\ \cdot & a_2 & a_1 & a_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

where $a_n \in \mathbb{C}$, meaning that

$$(Ax)_m = \sum_{n=-\infty}^{\infty} a_{m-n} x_n.$$

(a) Let $S : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ denote the right shift operator, defined by

$$(Sx)_m = x_{m-1}.$$

Show that a bounded linear operator on $\ell^2(\mathbb{Z})$ is a Laurent operator if and only if it commutes with S.

(b) Let $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ denote the unitary Fourier transform

$$\mathcal{F}f = \hat{f} \qquad \hat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-inx}\, dx.$$

Suppose that $M : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is the bounded multiplication operator

$$(Mf)(x) = a(x) f(x)$$

corresponding to multiplication by a function $a \in L^\infty(\mathbb{T})$. Show that

$$A = \mathcal{F} M \mathcal{F}^{-1}$$

is a Laurent operator whose matrix entries are the Fourier coefficients of a. What function $s(x)$ corresponds to S?

(c) Deduce that if a is nonzero, except possibly on a set of measure zero, and $1/a \in L^\infty(\mathbb{T})$, then the corresponding Laurent operator A is invertible. If

$$\left[A^{-1}\right] = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & b_0 & b_{-1} & b_{-2} & \cdot \\ \cdot & b_1 & b_0 & b_{-1} & \cdot \\ \cdot & b_2 & b_1 & b_0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

give an expression for the coefficients b_n in terms of a.