Problem Set 8: Math 201B

Due: Friday, March 4

1. A sequence of bounded linear operators $A_n \in \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is said to converge to an operator $A \in \mathcal{B}(\mathcal{H})$: *uniformly* if $A_n \to A$ with respect to the operator norm on $\mathcal{B}(\mathcal{H})$; *strongly* if $A_n x \to Ax$ strongly in $\mathcal{H}$ for every $x \in \mathcal{H}$; *weakly* if $A_n x \rightharpoonup Ax$ weakly in $\mathcal{H}$ for every $x \in \mathcal{H}$.

(a) Give an example of a sequence of operators that converges strongly but not uniformly.

(b) Give an example of a sequence of operators that converges weakly but not strongly.

2. A subset E of a (real or complex) vector space X is said to be convex if

$$\lambda x + (1-\lambda) y \in E \qquad \text{for every } x, y \in E \text{ and every } 0 \leq \lambda \leq 1.$$

(a) Show that a closed (*i.e.* strongly closed), convex subset of a Hilbert space is weakly closed.

(b) Show that every closed, convex subset of a Hilbert space contains a point of minimum norm.