Problem Set 9: Math 201B
Due: Friday, March 11

1. If $A : \mathcal{H} \to \mathcal{H}$ is a bounded, self-adjoint linear operator, show that

$$\|A^n\| = \|A\|^n$$

for every $n \in \mathbb{N}$. (You can use the results proved in class.)

2. Define $m : (0,1) \to (0,1)$ by

$$m(x) = \begin{cases} 0 & \text{if } 0 < x < 1/4, \\ 2(x - 1/4) & \text{if } 1/4 \leq x \leq 3/4, \\ 1 & \text{if } 3/4 < x < 1, \end{cases}$$

and let $M : L^2(0,1) \to L^2(0,1)$ be the multiplication operator $Mf = mf$. Determine the spectrum of M and classify it into point, continuous, and residual spectrum. Describe the eigenspace of any eigenvalues in the point spectrum.

3. Let $\{\lambda_n\}$ is a sequence of complex numbers such that $\lambda_n \to 0$ as $n \to \infty$ and define the operator $K : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by

$$K(x_1, x_2, \dots, x_n, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots, \lambda_n x_n, \dots).$$

(a) Prove that K is a compact operator. (Recall that a set is precompact if and only if it is totally bounded.)
(b) Let $P_n : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the orthogonal projection onto the nth component,

$$P_n(x_1, x_2, \dots, x_{n-1}, x_n, x_{n+1}, \dots) = (0, 0, \dots, 0, x_n, 0, \dots).$$

In what sense (uniformly, strongly, weakly) does the sum $\sum_{n\in\mathbb{N}} \lambda_n P_n$ converge to K? Does your answer change if $\lambda_n \not\to 0$ as $n \to \infty$?

4. Determine the spectra of the left and right shift operators on $\ell^2(\mathbb{N})$

$$S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots), \quad T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

and classify them into point, continuous, or residual spectrum.