Complex Analysis
Math 205A, Winter 2014
Final: Solutions

Part I: Short Questions

I.1 [5%] State the Cauchy-Riemann equations for a holomorphic function $f(z) = u(x, y) + iv(x, y)$.

Solution.

- The Cauchy-Riemann equations are

$$u_x = v_y, \qquad u_y = -v_x.$$

I.2 [5%] Give a formula for the radius of convergence R of the power series $\sum_{n=0}^{\infty} a_n z^n$ in terms of the coefficients a_n.

Solution.

- The Hadamard formula is

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}},$$

with the usual conventions for 0 and ∞.

I.3 [5%] State Cauchy's theorem in the most general form you know.

Solution.

- (a) The most general form of Cauchy's theorem that we proved in class is the homotopy version: If $f : \Omega \to \mathbb{C}$ is holomorphic in the open set $\Omega \subset \mathbb{C}$, and $\gamma_1, \gamma_2 : [a, b] \to \Omega$ are homotopic, piecewise-C^1 curves with the same endpoints, then

$$\int_{\gamma_1} f\, dz = \int_{\gamma_2} f\, dz.$$

In particular, if γ is a closed curve in Ω that is homotopic to a point, then

$$\int_{\gamma} f\, dz = 0.$$

- (b) The more general homology version of Cauchy's theorem states that $\int_\gamma f\,dz = 0$ if γ is homologous to zero in Ω (equivalent to the condition that the winding number $W_\gamma(c) = 0$ for every $c \in \mathbb{C} \setminus \Omega$).

I.4 [5%] Define the residue of a meromorphic function $f(z)$ at a pole z_0.

Solution.

- Here are three ways to define, or compute, the residue.

  (a) Let $\gamma(z_0)$ be a positively oriented, simple closed curve with z_0 in its interior such that f is holomorphic elsewhere in the interior of γ and on γ. Then

  $$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i}\int_{\gamma(z_0)} f(z)\,dz.$$

  (b) Suppose that the Laurent expansion of f at z_0 is

  $$f(z) = \sum_{n=-N}^{\infty} a_n(z-z_0)^n, \qquad 0 < |z - z_0| < R.$$

  Then $\operatorname{Res}(f, z_0) = a_{-1}$.

  (c) At a pole of order $N \geq 1$,

  $$\operatorname{Res}(f, z_0) = \frac{1}{(N-1)!}\lim_{z\to z_0}\frac{d^{N-1}}{d^{N-1}}\left[(z-z_0)^N f(z)\right].$$

  In particular, at a simple pole ($N = 1$),

  $$\operatorname{Res}(f, z_0) = \lim_{z\to z_0}\left[(z-z_0)f(z)\right].$$

Part II: Longer Questions

II.1 [20%] Suppose that $f(z)$ is an entire function such that

$$|f(z)| \le e^{\operatorname{Re} z} \qquad \text{for all } z \in \mathbb{C}.$$

Prove that $f(z) = ce^z$ for some constant $c \in \mathbb{C}$.

Solution.

- The function $f(z)/e^z$ is entire and bounded (since e^z is entire and non-zero, and $|f(z)| \le |e^z|$), so by Liouville's theorem it is equal to a constant.

II.2 [20%] (a) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{z^n}{1 - z^n} \right)$$

converges in $|z| < 1$ to a holomorphic function $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}$, and in $|z| > 1$ to a holomorphic function $g : \{z \in \mathbb{C} : |z| > 1\} \to \mathbb{C}$.
(b) Do you think that g can be obtained from f by analytic continuation across the unit circle $|z| = 1$?

Solution.

- (a) Let $|z| \le r$ where $0 < r < 1$. Then

  $$|1 - z^n| \ge 1 - |z|^n \ge 1 - r,$$

  and

  $$\left| \frac{1}{n^2} \left(\frac{z^n}{1 - z^n} \right) \right| \le \left(\frac{1}{1 - r} \right) \frac{r^n}{n^2}.$$

  The Weierstrass M-test implies that the series converges uniformly on $\bar{D}_r = \{z \in \mathbb{C} : |z| \le r\}$ by comparison with the convergent series $\sum r^n/n^2$. If K is a non-empty compact subset of the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$, then $\operatorname{dist}(K, \partial D) = \epsilon > 0$, so $K \subset \bar{D}_r$ for $r = 1 - \epsilon < 1$. Therefore, the series converges uniformly on compact subsets of D, and by Weierstrass's theorem the sum f is holomorphic in D.

- Similarly, if $|z| \geq R$ where $1 < R < \infty$. Then
$$\left|\frac{1}{z^n} - 1\right| \geq 1 - \frac{1}{|z|^n} \geq 1 - \frac{1}{R},$$
and
$$\left|\frac{1}{n^2}\left(\frac{z^n}{1-z^n}\right)\right| = \frac{1}{n^2}\left|\frac{1}{1/z^n - 1}\right| \leq \left(\frac{1}{1-1/R}\right)\frac{1}{n^2}.$$
The Weierstrass M-test implies that the series converges uniformly on $\bar{E}_R = \{z \in \mathbb{C} : |z| \geq R\}$ by comparison with $\sum 1/n^2$. If K is a non-empty compact subset of $E = \{z \in \mathbb{C} : |z| > 1\}$, then $K \subset \bar{E}_R$ for some $R > 1$, so the series converges uniformly on compact subsets of E, and the sum g is holomorphic in E.

- (b) The circle $|z| = 1$ is a natural boundary for both f and g, and one cannot obtain g by analytic continuation from f.

Remark. To prove the claim in (b), observe that if $z_0 = e^{i\theta}$ is a regular point on the boundary $|z| = 1$, meaning that f continues to a function that is holomorphic at z_0, then $\lim_{r\to 1^-} f(re^{i\theta})$ exists. Since the set of regular points is an open subset of the boundary, it is enough to show that there is a dense set of singular points where f has no analytic continuation. (Note that the divergence of the series at $z = z_0$ is not sufficient to imply that z_0 is a singular point.)

If $z_0 = e^{2\pi i p/q}$ is a qth root of unity, where q is prime and $1 \leq p \leq q-1$, and $0 < r < 1$, then
$$f(re^{2\pi i p/q}) = \sum_{n\equiv 0} \frac{1}{n^2}\left(\frac{r^n}{1-r^n}\right) + \sum_{k=1}^{q-1}\sum_{n\equiv k}^{\infty} \frac{1}{n^2}\left(\frac{r^n\omega^{pk}}{1-r^n\omega^{pk}}\right),$$
where $n \equiv k$ means modulo q, and $\omega = e^{2\pi i/q}$. Since $\omega^{pk} \neq 1$, we have
$$\frac{r^n}{1-r^n} \geq \left(\frac{1}{1-r}\right)\frac{r^n}{n}, \qquad \left|\frac{r^n\omega^{pk}}{1-r^n\omega^{pk}}\right| \leq \frac{1}{|1-\omega|},$$
so
$$\left|f(re^{2\pi i p/q})\right| \geq \frac{1}{1-r}\sum_{n\equiv 0}\frac{r^n}{n^3} - \frac{1}{|1-\omega|}\sum_{n\not\equiv 0}\frac{1}{n^2}.$$
It follows that $\left|f(re^{2\pi i p/q})\right| \to \infty$ as $r \to 1^-$, which proves that z_0 is a singular point.

II.3 [20%] (a) Define the Weierstrass $\wp$-function.
(b) The Korteweg-de Vries (KdV) equation is the following nonlinear partial differential equation for $u(x,t)$:

$$u_t + 6uu_x + u_{xxx} = 0.$$

Look for traveling wave solutions $u(x,t) = f(x-ct)$ of the KdV equation, where c is a constant, and find a differential equation satisfied by $f(\xi)$. Show that periodic traveling wave solutions can be expressed in terms of the $\wp$-function.
Hint. You can assume the differential equation for the $\wp$-function,

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

Solution.

- (a) Let $\omega_1, \omega_2 \in \mathbb{C} \setminus \{0\}$, with $\omega_2/\omega_1 \notin \mathbb{R}$, and let

  $$\Lambda = \{n_1\omega + n_2\omega_2 : n_1, n_2 \in \mathbb{Z}\}$$

  denote the associated lattice. Then

  $$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right].$$

- (b) Using $u = f(c-ct)$ in the KdV equation, we get the ODE

  $$-cf' + 6ff' + f''' = 0.$$

  One integration gives

  $$f'' + 3f^2 - cf + c_1 = 0,$$

  where c_1 is a constant of integration. Multiplying this equation by f' and integrating again, we get

  $$(f')^2 + 2f^3 - cf^2 + 2c_1f + c_2 = 0.$$

- The change of variable

$$f(z) = -2F(z) + \frac{c}{6},$$

  puts this equation in the form

$$(F')^2 = 4F^3 - g_2F^2 - g_3,$$

  where g_2, g_3 are constants (which depend on c, c_1, c_2).

- It follows that a solution is $F(z) = \wp(z+a)$, where a is an arbitrary constant, so

$$f(z) = -2\wp(z+a) + \frac{c}{6}.$$

- Alternatively, instead of integrating the ODE for f, you can differentiate the ODE for $\wp$.

Remark. The connection between the KdV equation and elliptic curves indicated here is the tip of a large iceberg. The KdV equation is a completely integrable soliton PDE and it turns out to have deep connections with algebraic geometry and Riemann surfaces.

II.4 [20%] Define a meromorphic function f on $\mathbb{C}$ by

$$f(z) = \frac{e^{-z^2/2}}{1+e^{-az}}, \qquad a = \sqrt{\pi}(1+i).$$

(a) Show that

$$f(z) - f(z+a) = e^{-z^2/2}.$$

(b) Use the residue theorem to evaluate

$$\int_{\gamma_R} f(z)\,dz,$$

where the contour γ_R is the rectangle with corners at $-R$, R, $R+i\sqrt{\pi}$, and $-R+i\sqrt{\pi}$. Assume that $R > \sqrt{\pi}/2$.
(c) Take the limit as $R \to \infty$ and deduce the value of the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}.$$

Solution.

- (a) We have

  $$a = \sqrt{2\pi}e^{i\pi/4}, \quad a^2 = 2\pi i, \quad e^{-a^2/2} = -1.$$

  It follows that

  $$\begin{aligned} f(z) - f(z+a) &= \frac{e^{-z^2/2}}{1+e^{-az}} - \frac{e^{-z^2/2-az-a^2/2}}{1+e^{-az-a^2}} \\ &= e^{-z^2/2}\left[\frac{1}{1+e^{-az}} + \frac{e^{-az}}{1+e^{-az}}\right] \\ &= e^{-z^2/2}. \end{aligned}$$

  (b) The function $f(z)$ has poles at points z such that $e^{-az} = -1$, or $az = (2n+1)\pi i$ for $n \in \mathbb{Z}$. Equivalently, $z = (2n+1)a/2$, and the only pole inside the contour γ_R is at $z = a/2$.

- Using l'Hôspital's rule, we find that $z = a/2$ is a simple pole with residue

$$
\begin{aligned}
\operatorname{Res}(f, a/2) &= \lim_{z \to a/2} \left[(z - a/2) \frac{e^{-z^2/2}}{1 + e^{-az}} \right] \\
&= e^{-a^2/8} \lim_{z \to a/2} \left[\frac{z - a/2}{1 + e^{-az}} \right] \\
&= e^{-i\pi/4} \lim_{z \to a/2} \left[\frac{1}{-ae^{-az}} \right] \\
&= \frac{e^{-i\pi/4}}{-ae^{-a^2/2}} \\
&= \frac{-i}{\sqrt{2\pi}}.
\end{aligned}
$$

The residue theorem implies that

$$
\int_{\gamma_R} f(z)\, dz = 2\pi i \operatorname{Res}(f, a/2) = \sqrt{2\pi}.
$$

- (c) We have

$$
\int_{\gamma_R} f(z)\, dz = \int_{-R}^{R} \left[f(x) - f(x + i\sqrt{\pi}) \right] \, dx + I_R,
$$

where

$$
I_R = i \int_0^{\sqrt{\pi}} \left[f(R + iy) - f(-R + iy) \right] \, dy.
$$

- For $0 \leq y \leq \sqrt{\pi}$ and $R \geq 2\sqrt{\pi}$, say, we estimate

$$
\begin{aligned}
|f(R + iy)| &\leq \frac{\left| e^{-(R+iy)^2/2} \right|}{1 - \left| e^{-a(R+iy)} \right|} \leq \frac{e^{-(R^2 - y^2)/2}}{1 - e^{-\sqrt{\pi}(R-y)}} \leq Ce^{-R^2/2}, \\
|f(-R + iy)| &\leq \frac{\left| e^{-(R-iy)^2/2} \right|}{\left| e^{a(R-iy)} \right| - 1} \leq \frac{e^{-(R^2 - y^2)/2}}{e^{\sqrt{\pi}(R+y)} - 1} \leq Ce^{-R^2/2}.
\end{aligned}
$$

Thus, the integrand in I_R converges uniformly to zero and $I_R \to 0$ as $R \to \infty$. It follows that

$$
\int_{-\infty}^{\infty} \left[f(x) - f(x + i\sqrt{\pi}) \right] \, dx = \lim_{R \to \infty} \int_{\gamma_R} f(z)\, dz = \sqrt{2\pi}.
$$

- Changing variables $x \mapsto x + \sqrt{\pi}$, we can write

$$\begin{aligned} \int_{-\infty}^{\infty} f(x + i\sqrt{\pi})\, dx &= \int_{-\infty}^{\infty} f(x + \sqrt{\pi}(1+i))\, dx \\ &= \int_{-\infty}^{\infty} f(x + a)\, dx. \end{aligned}$$

Using this result and (a) in the previous equation, we get that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \int_{-\infty}^{\infty} [f(x) - f(x+a)]\ dx = \sqrt{2\pi}.$$

Remark. Since $e^{-z^2/2}$ is an entire function, it's not obvious how one can use contour integration and the method of residues to evaluate the Gaussian integral. Ways to do this, like the one here, seem to have been discovered only as recently as the 1940s.

Part III: Extra Credit

III.1 [5%] Define the Riemann zeta function $\zeta(s)$. Prove that the only zeros of $\zeta(s)$ occur at $s = -2, -4, -6, \ldots$ or on the line $\mathrm{Re}\, s = 1/2$.

Solution.

- The Riemann zeta function is defined in the right-half plane Re$s > 1$ by the series
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$
which converges uniformly and absolutely on compact subsets of the half-plane to a holomorphic function.

- This function extends to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$; for example, by use of the functional equation
$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) = \Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s).$$